**ERGODIC** processes are stationary processes for which a time average from a single realisation can replace an ensemble average.

\[ \text{Mean } \mu_X(t, \tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} X_i(t, \tau) = \mu_X = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) \, dt \]

Auto-correlation \[ R_{xx}(t, \tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} X_i(t) X_i(t+\tau) = R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) x(t+\tau) \, dt \]

emphasize the many assumptions!

A sufficient but not necessary condition for ergodicity is

\[ \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |R_{xx}(\tau) - \mu_X^2| \, d\tau = 0 \]

Because of all these assumptions it is always good practice to write/talk about "estimates" of properties of a physical/biological system. An "estimate", however, implies both:

- probability of occurrence of an event
- uncertainty of the "estimate"; relate to probability tests for stationarity?
Recap prot where T was:
Definition
Ensemble

\{ Stationarity \} \text{ random processes }

Also, so far everything has been written down for continuous, infinite long records, however, all measurements are sort of continuous and of finite length \( \rightarrow \) possible biases and errors
\( \rightarrow \) statistical estimates, not exact values

Will return later \( \leftarrow \) RANDOM (STOCHASTIC) PROCESSES

2. DETERMINISTIC PROCESSES

(a) periodic, i.e., \( x(t) = x(t+T) \) of period \( T \)

\[ \begin{align*}
    x(t) &= x_0 \cos(\omega t + \phi) \\
         &= x_{01} \cos \omega t + x_{02} \sin \omega t
\end{align*} \]
(3) complex periodic: \[ x(t) = x(t + nT) \quad n = 1, 2, 3, \ldots \]

such data can always be written as a **Fourier Series**

\[
x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t)
\]

where \( f_0 = \frac{1}{2\pi T} \)

\[
a_n = \frac{2}{T} \int_{0}^{T} x(t) \cos(n \omega_0 t) \, dt \quad n = 1, 2, 3, \ldots
\]

\[
b_n = \frac{2}{T} \int_{0}^{T} x(t) \sin(n \omega_0 t) \, dt \quad n = 1, 2, 3, \ldots
\]

\[ \Delta f = f_0 = \frac{1}{T} \]
(c) almost periodic (not nonperiodic)

\[ x(t) = \sum_{i=1}^{\infty} A_i \cos(2\pi f_i t) \]

\( f_i \) are not rational numbers
\( N T \to \infty \)
incommensurable periods

discrete set of
Tides are of this form?

(d) transient dash
\[ x(t) = \begin{cases} A e^{-\alpha t} & t \geq 0 \\ \sigma & t < 0 \end{cases} \]

\[ x(t) = \begin{cases} A e^{-\alpha t} \cos \omega t & t \geq 0 \\ \sigma & t < 0 \end{cases} \]

\[ x(t) = \begin{cases} A \delta(t-b) & t \geq 0 \\ \sigma & t < 0 \end{cases} \]

and class 
\#2/99
CLASS #2

Fourier Transform

In 1807 Joseph Fourier announced that ANY periodic function \( x(t) = x(t+T) \) could be represented by the form

\[
x(t) = \sum_{i=0}^{\infty} a_i \cos\left(\frac{2\pi i t}{T}\right) + b_i \sin\left(\frac{2\pi i t}{T}\right)
\]

\[
= a_0 + \sum_{i=1}^{\infty} a_i \cos\left(\frac{2\pi i t}{T}\right) + b_i \sin\left(\frac{2\pi i t}{T}\right)
\]

APPLIES EVEN TO DISCONTINUOUS FUNCTIONS?

Leonard Luroth doubted this theorem. Dini and Diirchlet found that some mild restrictions are needed.

\[
\int_{-T/2}^{T/2} \cos\left(\frac{m 2\pi t}{T}\right) \cdot \cos\left(\frac{n 2\pi t}{T}\right) \, dt = \begin{cases} 0 & n \neq m \\ T/2 & m = n \geq 1 \\ 0 & m = n = 0 \end{cases}
\]

\[
\int_{-T/2}^{T/2} \sin\left(\frac{m 2\pi t}{T}\right) \cdot \sin\left(\frac{n 2\pi t}{T}\right) \, dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \geq 1 \\ \frac{T}{4} & m = n = 0 \end{cases}
\]

\[
\int_{-T/2}^{T/2} \sin\left(\frac{m 2\pi t}{T}\right) \cdot \cos\left(\frac{n 2\pi t}{T}\right) \, dt = 0
\]

\[
\int_{-T/2}^{T/2} x(t) \, dt = \frac{T}{2} a_0
\]

\[
\int_{-T/2}^{T/2} x(t) \cos\left(\frac{m 2\pi t}{T}\right) \, dt = a_m \cdot \frac{T}{2}
\]