CLASS #2

class #3/99

Fourier Transform

In 1807 Joseph Fourier anounced that ANY periodic function $x(t) = x(t+T)$ could be represented by the form

$$x(t) = \sum_{i=0}^{\infty} a_i \cos\left(i \frac{2\pi}{T} t\right) + b_i \sin\left(i \frac{2\pi}{T} t\right)$$

$$= a_0 + \sum_{i=1}^{\infty} a_i \cos(\ ) + b_i \sin(\ )$$

APPLIES EVEN TO DISCONTINEOUS FUNCTIONS !

orthogonal set of base functions to span a function space

$\langle \sin(nt) \cdot \sin(mt) \rangle = \delta_{nm}$

$\langle \cos(nt) \cdot \cos(mt) \rangle = \delta_{nm}$

$\langle \cos(nt) \cdot \sin(mt) \rangle = 0$

$\langle \cdot \rangle \equiv \frac{1}{T} \int \cdot \, dt$

Louis Lagrange doubted this theorem

Riemann and Dirichlet found that some mild restrictions are needed

$$\int_{-T/2}^{+T/2} \cos\left(m \frac{2\pi}{T} t\right) \cdot \cos\left(n \frac{2\pi}{T} t\right) dt = \begin{cases} 0 & n \neq m \\ T/2 & m = n \geq 1 \\ T & m = n = 0 \quad \text{to } a_0 \end{cases}$$

$$\int_{-T/2}^{+T/2} \sin\left(m \frac{2\pi}{T} t\right) \cdot \sin\left(n \frac{2\pi}{T} t\right) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \geq 1 \end{cases}$$

$$\int_{-T/2}^{T/2} \sin\left(m \frac{2\pi}{T} t\right) \cdot \cos\left(n \frac{2\pi}{T} t\right) dt = 0$$

$$\int_{-T/2}^{T/2} x(t) \sin\left(n \frac{2\pi}{T} t\right) dt = b_n \cdot \frac{T}{2}$$

$x(t)$

$$\text{to} \int_{-T/2}^{T/2} x(t)\, dt = \cancel{2}a_0 T \quad ; \quad \int_{-T/2}^{T/2} x(t) \cos\left(m \frac{2\pi}{T} t\right) dt = a_m \cdot \frac{T}{2}$$

or

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos\left(n\frac{2\pi}{T}t\right)dt$$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\sin\left(n\frac{2\pi}{T}t\right)dt$$

another (convenient) way to write this is

$$x(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi}{T}t\right) + b_n \sin\left(n\frac{2\pi}{T}t\right) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt}$$

$j = \sqrt{-1}$,

$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$

$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$

$e^{j\theta} = \cos\theta + j\sin\theta$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt$$

or

$$x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \overbrace{\int_{-T/2}^{T/2} x(q)\, e^{-j\frac{2\pi}{T}nq}\, dq}^{c_n}\; e^{j\frac{2\pi}{T}nt}$$

Next we want to apply this "result" to non-periodic functions; we interpret these as periodic functions with a period $T \to \infty$

$\rightarrow \quad \frac{1}{T} = \Delta f \qquad \frac{n}{T} = f = n \cdot \Delta f \qquad f$ - frequency

$$x(t) = s(t) \equiv \lim_{\substack{T \to \infty \\ \Delta f \to df \\ n\Delta f \to f}} \sum_{n=-\infty}^{\infty} \Delta f \int_{-T/2}^{T/2} s(q)\, e^{-j\frac{2\pi}{T} n q}\, dq\; e^{j2\pi\frac{n}{T}t}$$

hand waving #1 because exchange of Σ with $\int$

$$= \int_{-\infty}^{\infty} df \underbrace{\int_{-\infty}^{\infty} s(q)\, e^{-j2\pi f q}\, dq}_{S(f)}\; e^{j2\pi f t}$$

more hand waving because of taking the limits

$$s(t) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f t}\, df$$

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f t}\, dt$$

These are Fourier transform pairs; valid if

(1) all integrals converge

(2) $\lim_{|t| \to \infty} s(t) = 0$

(3) $\int_{-\infty}^{\infty} |s(t)|^2\, dt = $ finite ("energy")

class #2 9/8/95