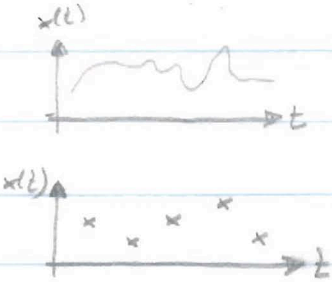
CLASS #3

record length T / time step Δt	infinitely long record	finite record
continuous sampling	analytical	leakage
discrete sampling	aliasing	aliasing + leakage



How can we force the discrete sampling and finite record length into the framework of Fourier Transforms?

→ DISCRETE FT

Which effects ~~does~~ and artifacts does the DISCRETE FT ~~have~~ introduce?

→ leakage and aliasing

finite T finite Δt

In order to discuss these artifacts and to minimize them and to design proper experiments involving temporal and/or spatial data collection schemes, we need to first discuss generalized functions such as the δ functions.

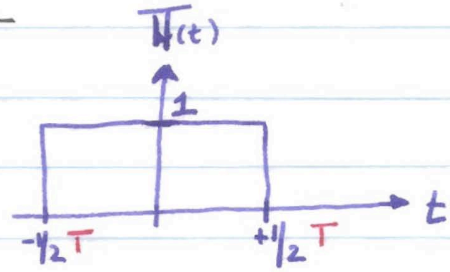
or Dirac

how multiplications in the time domain translate into the frequency domain.

Leakage

~~Generalized (δ) functions~~

example of Fourier Transform



look at this as a data "window"

$$S'(f) = \int_{-\infty}^{\infty} \Pi(t)\, e^{-j 2\pi f t}\, dt$$

need: $e^{j\theta} = \cos\theta + j \sin\theta$

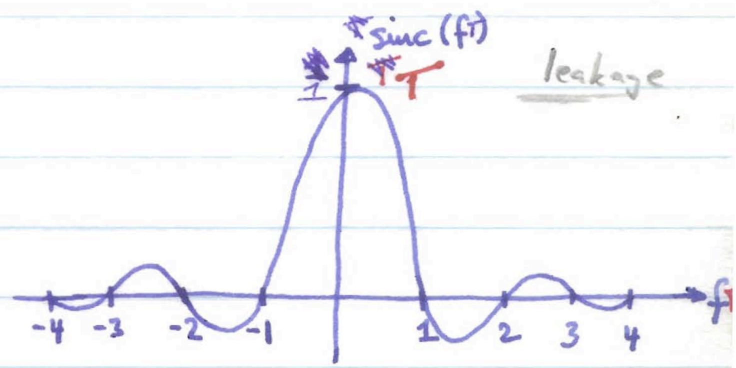
$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$

$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$

$$= \int_{-\frac{1}{2}T}^{\frac{1}{2}T} e^{-j 2\pi f t}\, dt$$

$$= T \frac{1}{2\pi j f T}\left[e^{j\pi f T} - e^{-j\pi f T}\right]$$

$$= T \frac{\sin(\pi f T)}{\pi f T}$$



leakage

$f' \equiv fT$ $\quad \equiv T \,\mathrm{sinc}\,(fT)$

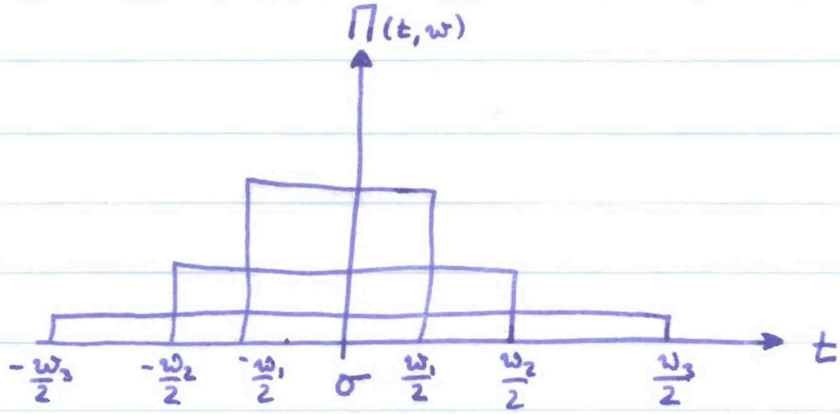
$$\lim_{f' \to 0} \frac{\sin \pi f'}{\pi f'} = \lim_{f' \to 0} \frac{\pi \cos \pi f'}{\pi} = \lim_{f' \to 0} \cos \pi f' = 1$$

L'Hospital

Built a sequence of gate functions $\Pi(t,w)$ that have variable width w but uniform area, i.e.,

$$\Pi(t,w) = \begin{cases} 1/w & -w/2 \le t \le w/2 \\ 0 & |t| > w/2 \end{cases}$$

$$\text{area} = \int_{-\infty}^{\infty} \Pi(t,w)\, dt = 1$$



Now

$$\lim_{w \to 0} \int_{-\infty}^{\infty} \Pi(t,w)\, x(t)\, dt = \lim_{w \to 0} \frac{1}{w} \int_{-w/2}^{w/2} x(t)\, dt$$

$$= \lim_{w \to 0} \frac{1}{w}\, x(\xi) \cdot w \qquad \xi \in [-w/2, w/2]$$

$$= x(0)$$

Define this sequence as $\int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt = x(t=0)$ sifting

mechanism to pick out a discrete data point from a process through a measurement i.e.

$$\delta(t) = \begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases} \qquad \text{with} \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\delta(t) = \lim_{w\to 0} \Pi(t, w)$$

$$s(t) = \begin{cases} 1 & |t|\le T/2 \\ 0 & |t| > T/2 \end{cases}$$

$\Downarrow$ $S(f) = T\cdot \mathrm{sinc}(fT)$

as $T \to \infty$ for $s(t)$
$S(f) \to \delta(f)$ $\longrightarrow$

$$\delta(f) = \lim_{w\to\infty} \int_{-\infty}^{\infty} \Pi(t,w)\, e^{-j2\pi f t}\, dt$$

discuss these properties with the sinc curve

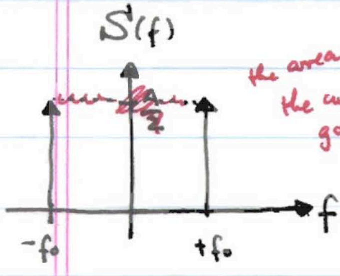
We will need these functions to evaluate the Fourier transforms of periodic functions such as

$$s(t) = A \cos 2\pi f_0 t$$

i.e.,

$$S(f) = \mathcal{F}(s(t)) = \int_{-\infty}^{\infty} a \cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt$$

$$= \frac{A}{2} \int_{-\infty}^{\infty} \left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) e^{-j2\pi f t}\, dt$$



the area under the curve is going to be A|2

$$= \frac{A}{2} \int_{-\infty}^{\infty} \underbrace{1\cdot e^{-j2\pi(f-f_0)t}}_{\text{FT of } x(t)=1} + \underbrace{1 e^{-j2\pi(f+f_0)t}}_{\text{FT of } x(t)=1}\, dt$$

$$= \frac{A}{2}\, \delta(f-f_0) + \frac{A}{2}\, \delta(f+f_0)$$