Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\; x(t-\tau)\; d\tau \equiv h * x$$

a window of weights moved over the data   data

look at it like an operation that involves a "running mean" where $h(\tau)$ represents some weighting function ~~and~~ that varies at a lag time τ and some data $x(t-\tau)$ collected at a ~~past~~ time $(t-\tau)$. The "running mean" is over the length of the filter, infinite as the limits of the integral indicates

Convolution Theorem says that if

$$Y(f) = \mathcal{F}(y(t))$$
$$H(f) = \mathcal{F}(h(t))$$
$$X(f) = \mathcal{F}(x(t))$$

$$\mathcal{F}(\cdot) \equiv \int_{-\infty}^{\infty} \cdot\; e^{-j2\pi f t}\, dt$$

and

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \equiv h * x$$

then

$$Y(f) = H(f) \cdot X(f)$$

A ~~multiplication~~ convolution in the time domain of two time series corresponds to a multiplication in the frequency domain (the reverse holds also, i.e., a multiplication in the time domain corresponds to a convolution in the frequency domain.

does not depend on t — does not depend on τ

Proof.: $$Y(f) = \underbrace{\int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau}_{g(t)} \; e^{-j2\pi f t}\, dt}_{\mathcal{F}(g(t))}$$

switched order of integration

$$= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} x(t-\tau)\, \cancel{dt}\ \cancel{d\tau}\ e^{-j2\pi f t}\, dt\, d\tau$$

$\sigma = t - \tau$

$\frac{d\sigma}{dt} = 1$

$$= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} x(\sigma)\, e^{-j2\pi f(\sigma+\tau)}\, d\sigma\, d\tau$$

$\sigma = t - \tau$

$$= \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f\tau} \underbrace{\int_{-\infty}^{\infty} x(\sigma)\, e^{-j2\pi f\sigma}\, d\sigma}_{\cancel{X}\ \mathcal{F}(x(t-\tau)) = X(f)}\, d\tau$$

does not depend on σ or t or τ

$$= X(f) \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau}_{\mathcal{F}(h(\tau)) = H(f)}$$

$$= H(f) \cdot X(f)$$

Properties of convolution

(i) $h * x = x * h$ commutative

(ii) $h * (a * x) = (h * a) * x$ associative

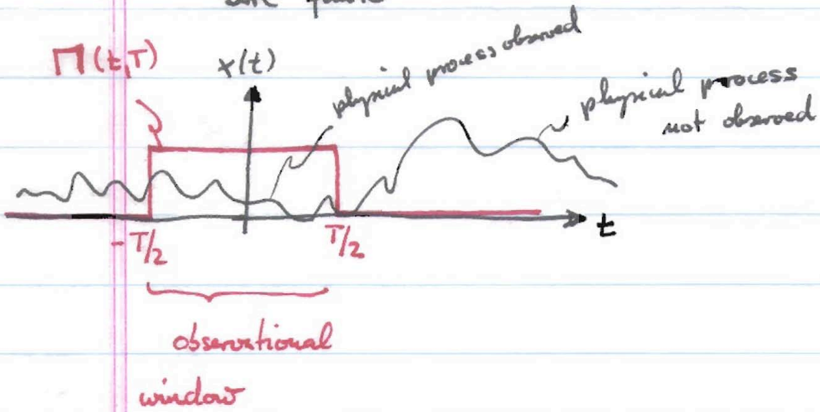
(iii) $h * (x_1 + x_2) = h * x_1 + h * x_2$ distributative with respect to addition

class #3

(iv) $\mathcal{F}(h * x) = H \cdot X$ convolution theorem

Record length

Now, apply the convolution theorem to data that are continuous but are finite



$$x(t) = \cos 2\pi f_0 t$$

$$\hat{x}(t,T) = \cos(2\pi f_0 t) \cdot \Pi(t,T)$$

$$\mathcal{F}(\Pi(t,T)) = T \,\mathrm{sinc}(fT) = T \,\frac{\sin(\pi f T)}{\pi f T}$$

$\cancel{\mathcal{F}(\hat{x}(t,T)) =}$

$$\mathcal{F}(\hat{x}(t,T)) = \mathcal{F}(x(t)) * \overset{\mathcal{F}(\Pi(t,T))}{\cancel{T \operatorname{sinc}(fT)}}$$

$$= X(f) * T \cdot \operatorname{sinc}(fT)$$

$$= \int_{-\infty}^{\infty} T \operatorname{sinc}(f'T) \; X(f-f') \, df'$$

$$= \int_{-\infty}^{\infty} T \operatorname{sinc}(f'T) \; \frac{1}{2} \left[\delta(f+f_0-f') + \delta(f-f_0-f') \right] df'$$

$$= \frac{T}{2} \left\{ \operatorname{sinc}[(f+f_0)T] + \operatorname{sinc}[(f-f_0)T] \right\}$$

Compare this result with the earlier one, i.e., on p. 14

$$\mathcal{F}(x(t)) = \frac{1}{2} \left\{ \delta(f+f_0) + \delta(f-f_0) \right\}$$

this is referred to as leakage

$$\Rightarrow \quad \mathcal{F}(x(t)\,\Pi(t,T)) \neq \mathcal{F}(x(t))$$

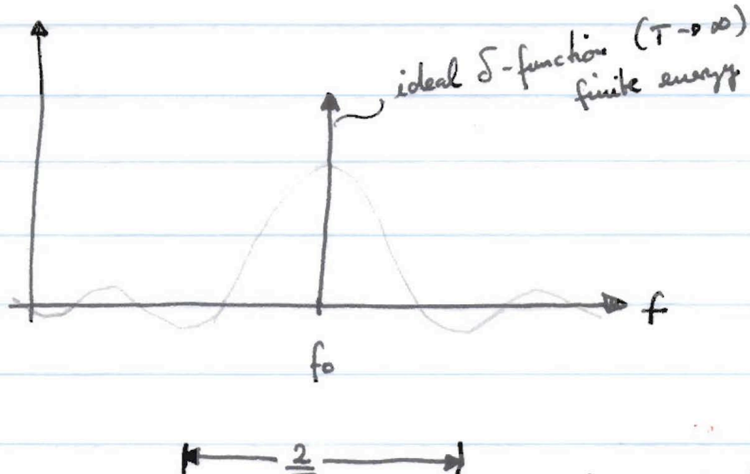
But

$$\lim_{T \to \infty} T \operatorname{sinc}[(f+f_0)T] = \delta(f+f_0)$$

~~Useful~~

The function $\quad \text{sinc}\, fT = \dfrac{\sin(\pi f T)}{\pi f T}$

needs to be made wide in the time domain $(T \to \infty)$ in order to reflect the narrow δ-function like character in the frequency domain (little leakage, little sidelobes of the sinc fct.)

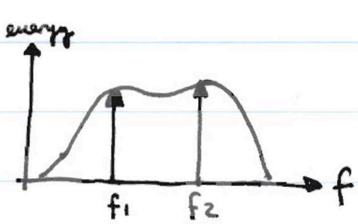


width of the 1^{st} sidelobe 93% of energy

1. resolution problem arises as the sidelobes of two sinc fctn run into each other

2. false "mean" or "zero frequency" component

3. two frequencies may overlap



$(f_2 - f_1) < \frac{2}{T}$

But for $A_1 \gg A_2$: