6.1 (20)

put fig. ~~5.3~~ of Brigham (1988) on the board as an "outline"

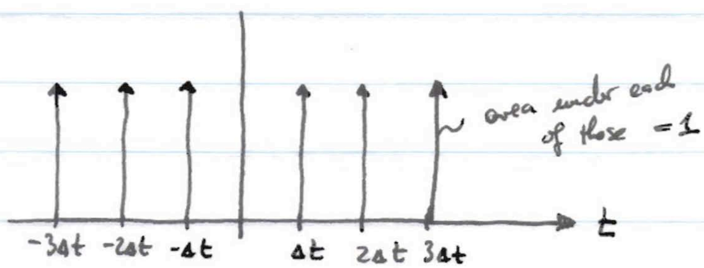
Sampling : discrete sampling → aliasing

Define a "sampling" function

$$Ш(\Delta t, t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) = \begin{cases} \infty & t = n\Delta t \\ 0 & t \neq n\Delta t \end{cases} \qquad n = 0, \pm 1, \pm 2, \ldots$$

a "data comb"

a train of unit impulse functions



We can now define a sampled or discrete version $\hat{x}(t, \Delta t)$ of the continuous process $x(t)$ as

$$\hat{x}(t, \Delta t) = Ш(t, \Delta t) \cdot x(t)$$

This multiplication in the time domain will correspond to a convolution in the frequency domain that ~~we'll discuss~~ is the main goal of this lecture ⟶ SAMPLING THEOREM

First, however, we need to explore $Ш(t, \Delta t)$.

Interpret $\text{III}(t,\Delta t)$ as a periodic function with period Δt
Its Fourier series representation is thus

$$\text{III}(t,\Delta t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j 2\pi \frac{n}{\Delta t} t}$$

with

$$c_n = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} \text{III}(\Delta t, t) \, e^{-j \frac{2\pi}{\Delta t} n t} \, dt$$

$t \in \left[-\frac{\Delta t}{2}, \frac{\Delta t}{2}\right]$ $\text{III}(\Delta t, t)$ on that interval equals $\delta(t)$

$$= \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} \delta(t - n\Delta t) \, e^{-j \frac{2\pi}{\Delta t} n t} \, dt$$

$$= \frac{1}{\Delta t} \, e^{-j \frac{2\pi}{\Delta t} n t} \Big|_{t = n\Delta t} = \frac{1}{\Delta t} \, e^{-j 2\pi n^2}$$

$$= \frac{1}{\Delta t} \cos(2\pi n^2) - i \sin(2\pi n^2) = \frac{1}{\Delta t}$$

Hence

$$\text{III}(t,\Delta t) = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} e^{j 2\pi \frac{n}{\Delta t} t}$$

The Fourier Transform of this sampling function thus is

$$\mathcal{F}(\text{III}(t,\Delta t)) = \frac{1}{\Delta t} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j 2\pi \frac{n}{\Delta t} t} \, e^{-j 2\pi f t} \, dt$$

$$\mathcal{F}(\mathrm{III}(t,\Delta t)) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\frac{2\pi}{\Delta t}nt}\cdot e^{-j2\pi ft}\,dt$$

$$\stackrel{*}{=} \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} 1\cdot e^{-j2\pi f't}\,dt \qquad f' = f-\frac{n}{\Delta t}$$

$$= \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty}\mathcal{F}(1) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty}\delta(f') = \frac{1}{\Delta t}\sum_{n=-\infty}^{n=+\infty}\delta\left(f-\frac{n}{\Delta t}\right)$$

This is quite a remarkable and very useful result, i.e., the Fourier transform of a sequence of δ-functions is a sequence of δ-functions

$$\mathcal{F}\left(\sum_{n=-\infty}^{\infty}\delta(t-n\Delta t)\right) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{\Delta t}\right)$$

Now we can compute the Fourier Transform of our sampled time series $\hat{x}(t,\Delta t) = \mathrm{III}(t,\Delta t)\cdot x(t)$:

$$\mathcal{F}(\hat{x}(t,\Delta t)) = \mathcal{F}(\mathrm{III}(t,\Delta t)) * \mathcal{F}(x(t))$$

$$= \int_{-\infty}^{\infty}\tilde{\mathcal{F}}(\mathrm{III}(f'))\cdot X(f-f')\,df' \qquad \text{convolution}$$

$$\stackrel{**}{=} \sum_{n=-\infty}^{\infty}\frac{1}{\Delta t}\int_{-\infty}^{\infty}\delta(f-n/\Delta t)\cdot X(f-f')\,df'$$

$$\mathcal{F}(\hat{x}(t,\Delta t)) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} X(f-f') \qquad \text{with } f' = n/\Delta t$$

as then $\delta(f' - \frac{n}{\Delta t}) = \delta(0)$

$$\boxed{\mathcal{F}(\hat{x}(t,\Delta t)) = \frac{1}{\Delta t}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{\Delta t}\right) \neq X(f) = \mathcal{F}(x(t))}$$

the sampled waveform is an infinitely shifted sum of the contineous (unsampled) Fourier transform $X(f)$. Instead of a single transformed waveform we now have an infinite number of these which possibly overlap! The overlap or separation is crucially dependent upon Δt which is the sampling intervall. The potentially devastating effect of the discrete sampling is called <u>ALIASING</u>.

Discuss graphically (Bingham, 1988; fig. 5.3-5.8) the

class #4 diffrent choices of Δt and what they'll do to the Fourier transform of your data.

How does one ~~made~~ avoid aliasing? not possible in practice but ...

(1) Data may not contain any contributions from frequencies larger than f_N, the Nyquist frequency
→ data with a limited frequency band width

(2) Data must be sampled at a rate faster than $f_N = 1/2\Delta t$