where f_N is the highest frequency present in the data

go to page 25

SAMPLING THEOREM

(a) time)

If $\mathcal{F}(x(t)) = \begin{cases} X(f) \neq 0 & |f| \leq f_N \\ 0 & |f| \geq f_N \end{cases}$ band-limited

then

$$\hat{x}(t, \Delta t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\, \delta(t - n\Delta t)$$

with
~~where~~ $\Delta t = 1/2f_N$ can be used to uniquely determine

the continuous fctn.
~~In particular~~ $x(t)$ ~~is then given by~~

$$x(t) = \Delta t \sum_{n=-\infty}^{\infty} x(n\Delta t) \underbrace{\frac{\sin 2\pi f_N (t - n\Delta t)}{\pi (t - n\Delta t)}}_{\frac{\sin(\Delta t \cdot \alpha)}{\alpha}}$$

$\alpha = \pi(t \cdot n\Delta t)$

(b) frequency

If $x(t) = \begin{cases} 0 & |t| \geq T \\ \neq 0 & t < T \end{cases}$

go to p. 25

then

$$\mathcal{F}(x(t)) = \sum_{n=-\infty}^{\infty} X(n/T)\, \delta(f - n/T)$$

where $\Delta f = 1/T$

Class #5 review concept of aliasing again and careful

SAMPLING THEOREM

If

$$\mathcal{F}(x(t)) = \begin{cases} X(f) & \cancel{|f| \leq f_c} \quad |f| < f_c \\ 0 & \cancel{|f| > f_c} \quad |f| \geq f_c \end{cases}$$ band-limited wave form

then the continuous function $x(t)$ can be determined uniquely from its sampled values

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n \cdot \Delta t)\, \delta(t - n \cdot \Delta t)$$

where

$$\Delta t = 1/2f_c \quad \text{or} \quad f_c = 1/(2\Delta t)$$

and f_c is the Nyquist sampling frequency. Specifically, $x(t)$ is given as

$$x(t) = \Delta t \cdot \sum_{n=-\infty}^{\infty} x(n \cdot \Delta t) \cdot \frac{\sin\left(2\pi f_c (t - n\Delta t)\right)}{\pi (t - n\Delta t)}$$

$$= \sum_{n=-\infty}^{\infty} x(n \cdot \Delta t) \cdot \underbrace{\frac{\sin[\pi (t - n\Delta t)/\Delta t]}{\pi (t - n\Delta t)/\Delta t}}_{\operatorname{sinc}(t^*)} \qquad t^* = \frac{\pi (t - n\Delta t)}{\Delta t}$$

(need proof; graphical)
Bracewell Fig. 5.7

The conditions above make explicitly sure that no aliasing occurs, i.e., without aliasing we can construct a continuous band-limited function with discrete samples that are $\Delta t / 2f_c$ apart. Aliasing is avoided if and only if

(1) the time series is band-limited, i.e., $\mathcal{F}(x(t)) = 0$ for $|f| \leq f_c$

(2) the time series is sampled with a time step $\Delta t \leq 1/2f_c$

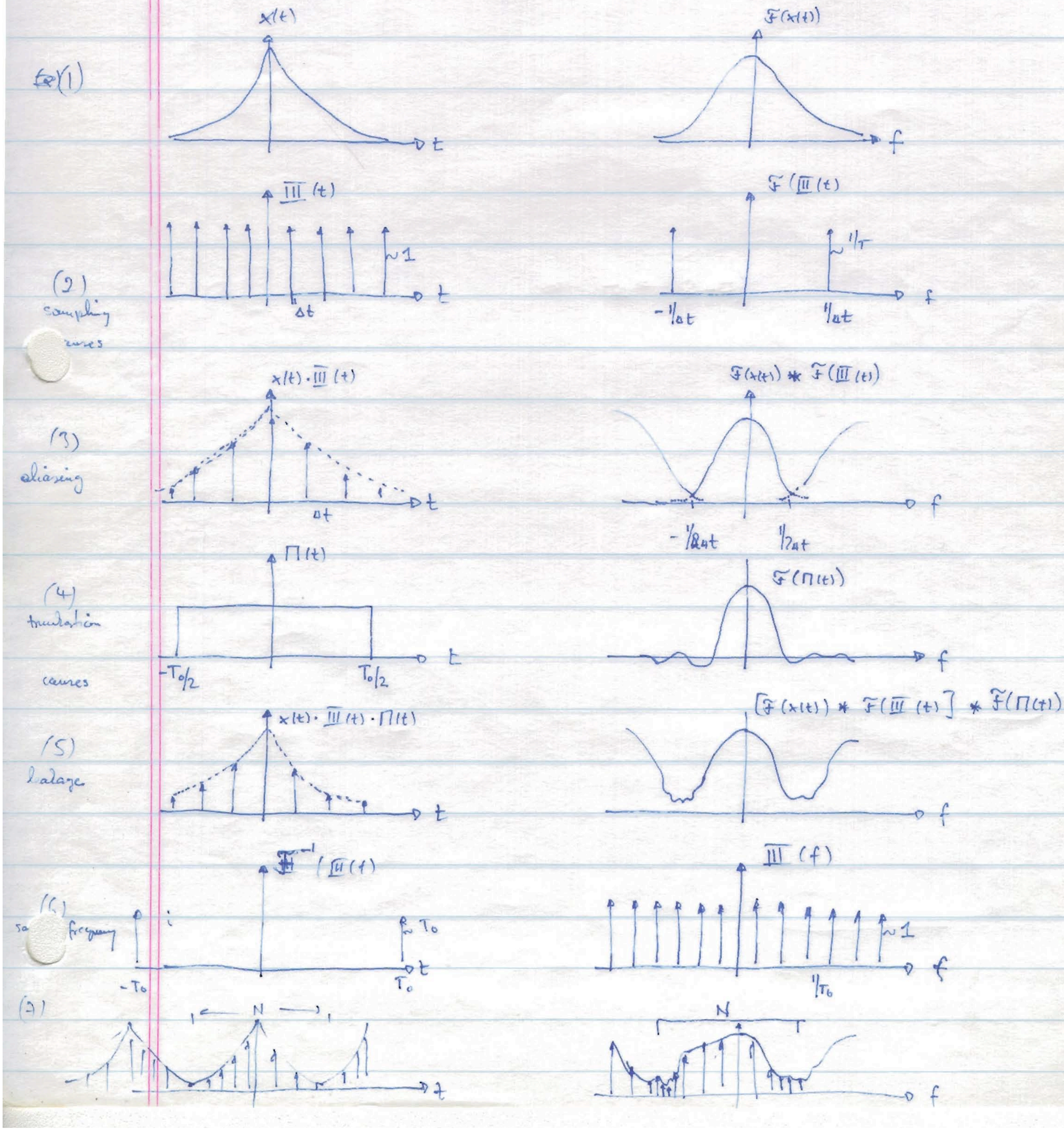
[(3) the sampled time series is infinitely long]

Discrete Fourier Transform

Brigham Fig. 6.1

How does the discrete (digital) Fourier transform relate to the continuous Fourier transform?

Derive discrete FT as a special case of the continuous FT



DISCRETE FOURIER TRANSFORM

Brigham (1974) chapt. 6

need to modify the continuous Fourier transform in order to make it acceptable for machine (discrete) computations, i.e., both time and frequency fctn are periodic ~~about T_0 and $1/2\Delta t$~~, respectively. Modifications are

(a) time domain sampling ...

(b) time domain truncation ...

(c) frequency domain sampling ...

(a) $$x(t)\cdot Ш(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-k\Delta t) = \sum_{k=-\infty}^{\infty} x(k\Delta t)\,\delta(t-k\Delta t)$$

(b) $$x(t)\cdot Ш(t)\cdot \Pi(t) = \left[\sum_{k=-\infty}^{\infty} x(k\Delta t)\,\delta(t-k\Delta t)\right]\cdot \Pi(t) = \sum_{k=0}^{N-1} x(k\Delta t)\,\delta(t-k\Delta t))$$

(c) $$\left[x(t)\cdot Ш(t)\cdot\Pi(t)\right] * \mathcal{F}^{-1}(Ш(f)) = \left[\sum_{k=0}^{N-1} x(k\Delta t)\,\delta(t-k\Delta t)\right] * \left[T_0\sum_{r=-\infty}^{\infty}\delta(t-rT_0)\right]$$

$$= \ldots + T_0\sum_{k=0}^{N-1} x(k\Delta t)\,\delta(t+T_0-k\Delta t)$$

$$+ T_0\sum_{k=0}^{N-1} x(k\Delta t)\,\delta(t-k\Delta t)$$

$$+ T_0\sum_{k=0}^{N-1} x(k\Delta t)\,\delta(t-T_0-k\Delta t) + \ldots$$

this is a periodic function with period T_0 expressed by N samples (sampling theorem)

$$= T_0\sum_{r=-\infty}^{\infty}\sum_{k=0}^{N-1} x(k\Delta t)\,\delta(t-k\Delta t+rT_0)$$

Now we need to take the Fourier Transform of this mess

The time domain sampled and truncated and

frequency domain sampled function

$$\left[x(t) \cdot Ш(t) \cdot \Pi(t)\right] * \tilde{\mathcal{F}}^{-1}\left(Ш(f)\right)$$

$$= T_0 \sum_{r=-\infty}^{+\infty} \sum_{k=0}^{N-1} x(k\Delta t)\, \delta(t - k\Delta t + r T_0)$$

This is a periodic function with period T_0 that is expressed by N samples. We will need the Fourier Transform of this construct $\longrightarrow$ Discrete Fourier Transform

First recall, however, that a PERIODIC function has a Fourier Series representation

$$x(t) = \sum_{n=\infty}^{+\infty} c_n\, e^{j\frac{2\pi}{T_0} n t} \qquad \text{with } c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x(t)\, e^{-j\frac{2\pi}{T_0} n t}\, dt$$

The Fourier Transform of this Fourier Series is

$$X(f) = \int_{-\infty}^{+\infty} \sum_{n=\infty}^{\infty} c_n e^{+j\frac{2\pi}{T_0} n t} \cdot e^{-j 2\pi f t}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} c_n \int_{-\infty}^{+\infty} e^{j 2\pi\left(\frac{n}{T_0} - f\right)t}\, dt$$

Rewrite exponential and interpret the integral as the Fourier Transform of 1 :

$$X(f) = \sum_{n=-\infty}^{+\infty} c_n \int_{-\infty}^{+\infty} 1 \cdot e^{-j\,2\pi(f-\frac{n}{T_0})t}\, dt$$

(*) consider dimensions or units of $\delta(f)$

$$= \sum_{n=-\infty}^{+\infty} c_n \underbrace{\delta(f-\tfrac{n}{T_0})}_{\text{dimensions of } (\cancel{\text{time}}) = (\text{frequency})^{-1}} \cancel{\Delta t}$$

This is a continuous Fourier Transform for all frequencies f

Look at discrete samples of this, say, at

$$f = n/T_0$$

then

$$X(f=\tfrac{n}{T_0}) = \Delta t \cdot c_n$$

The $\delta(f-\frac{n}{T_0})$ picks out the value for $f = n/T_0$ which here is c_n

So, all we have left to do is to find the Fourier Series coefficients $c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} y(t)\, e^{-j 2\pi \frac{n}{T_0} t}\, dt$

where $y(t) = T_0 \sum_{r=-\infty}^{+\infty} \sum_{k=0}^{N-1} x(k\Delta t)\cdot \delta(t-k\Delta t + rT_0)$

$$c_n = \frac{1}{T_0} \int_{-\frac{\Delta t}{2}}^{T_0 - \frac{\Delta t}{2}} \left\{ [x(t) \cdot Ш(t) \cdot \Pi(t)] * \tilde{\mathcal{F}}^{-1}(Ш(f)) \right\} e^{-j 2\pi \frac{n}{T_0} t} \, dt$$

$$= \frac{1}{T_0} \int_{-\frac{\Delta t}{2}}^{T - \frac{\Delta t}{2}} \left\{ T_0 \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} x(k\Delta t) \, \delta(t - k\Delta t + r T_0) \right\} e^{-j 2\pi \frac{n}{T_0} t} \, dt$$

Note that the integral is evaluated only for

$$t \in \left[-\frac{\Delta t}{2} \, , \, T_0 - \frac{\Delta t}{2} \right]$$

thus there is only <u>one</u> Delta-function active in this intervall, e.g., only $r = 0$ contributes because

$$\delta(t - k\Delta t + r T_0)$$

is zero for all $r \neq 0$

$$\Downarrow \quad c_n = \sum_{k=0}^{N-1} \underbrace{x(k\Delta t)}_{\text{does not depend on } t} \underbrace{\int_{-\Delta t/2}^{+T_0 - \Delta t/2} \delta(t - k\Delta t) \, e^{-j 2\pi \frac{n}{T_0} t} \, dt}_{= e^{-j 2\pi \frac{n}{T_0} t} \Big|_{t = k\Delta t}}$$

$$\Downarrow \quad c_n = \sum_{k=0}^{N-1} x(k\Delta t) \, e^{-j 2\pi \frac{n}{T_0} k\Delta t} \qquad \text{and } T_0 = N \cdot \Delta t$$

$$c_n = \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j 2\pi \frac{n}{N\cancel{\Delta t}} k \cancel{\Delta t}} \qquad N \cdot \Delta t = T_0$$

and

$$X(f = \tfrac{n}{T_0}) = \Delta t \; c_n$$

or

$$\boxed{X\left(f = \frac{n}{N\Delta t}\right) = \Delta t \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j 2\pi \frac{n \cdot k}{N}}}$$

This remarkably "simple" result is the Discrete Fourier Transform

It maps a set of N numbers x_k (time domain)

into a set of N numbers X_n (frequency domain)

Perhaps somewhat surprising, but consider

$$n = r \; : \; X(f = \tfrac{r}{T_0}) = \Delta t \sum_{k=0}^{N-1} x_k \, e^{-j 2\pi \frac{r \cdot k \cdot \Delta t}{T_0}}$$

$$n = r + N \; : \; X\left(f = \frac{r+N}{T_0}\right) = \Delta t \sum x_k \, \underbrace{e^{-j 2\pi \frac{(r+N) k \Delta t}{T_0}}}_{e^{-j 2\pi \frac{r k \Delta t}{T_0}} \cdot \underbrace{e^{-j 2\pi \frac{N \cdot \Delta t}{T_0}}}}$$

$$\Downarrow$$

$$X(f = \tfrac{r}{T_0}) = X\left(f = \tfrac{r+N}{T_0}\right) \qquad e^{-j 2\pi} = 1$$

There are only N distinct values of $X(f = \frac{n}{T_0})$

Appendix to 28-2 and 28-3

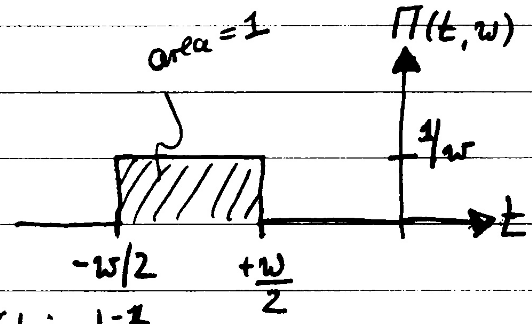
Units of $\delta(t)$:

$$x(t_0) \equiv \int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt$$

$\Rightarrow$ $\delta(t)$ has units of $(\text{time})^{-1}$

Alternatively

$$\delta(t) = \lim_{w \to 0} \Pi(t,w)$$



$\Rightarrow$ $\delta(t)$ has units of $\Pi(t,w)$ which is $(\text{time})^{-1}$

Units of $\delta(f)$:

$$X(f_0) = \int_{-\infty}^{+\infty} X(f)\cdot\delta(f-f_0)\,df$$

$\Rightarrow$ $\delta(f)$ has units of $(\text{time}) = (\text{frequency})^{-1}$

and

$$\sum_{n=-\infty}^{+\infty} c_n\,\delta\!\left(f-\frac{n}{T_0}\right) = \Delta t \cdot c_n \qquad \text{for } f = \frac{n}{T_0}$$

$$\text{or } n = f T_0$$

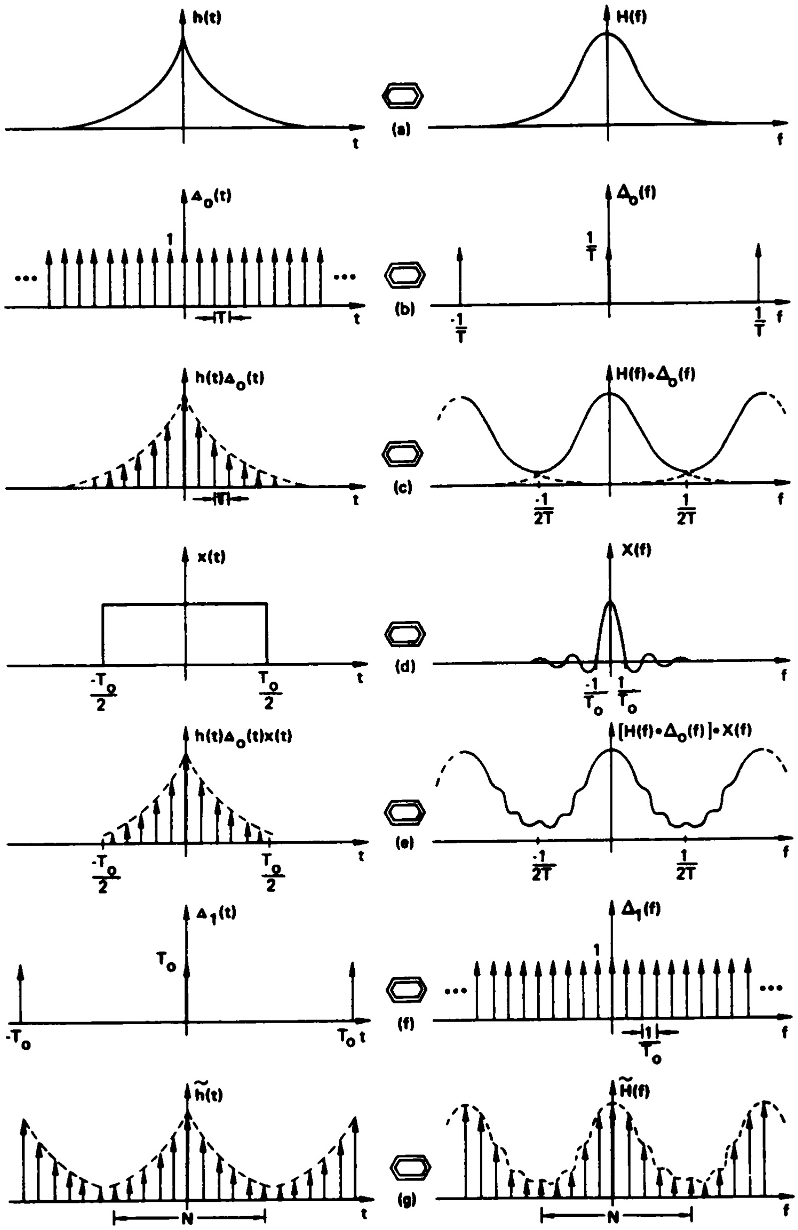


Figure 6.1 Graphical development of the discrete Fourier transform.

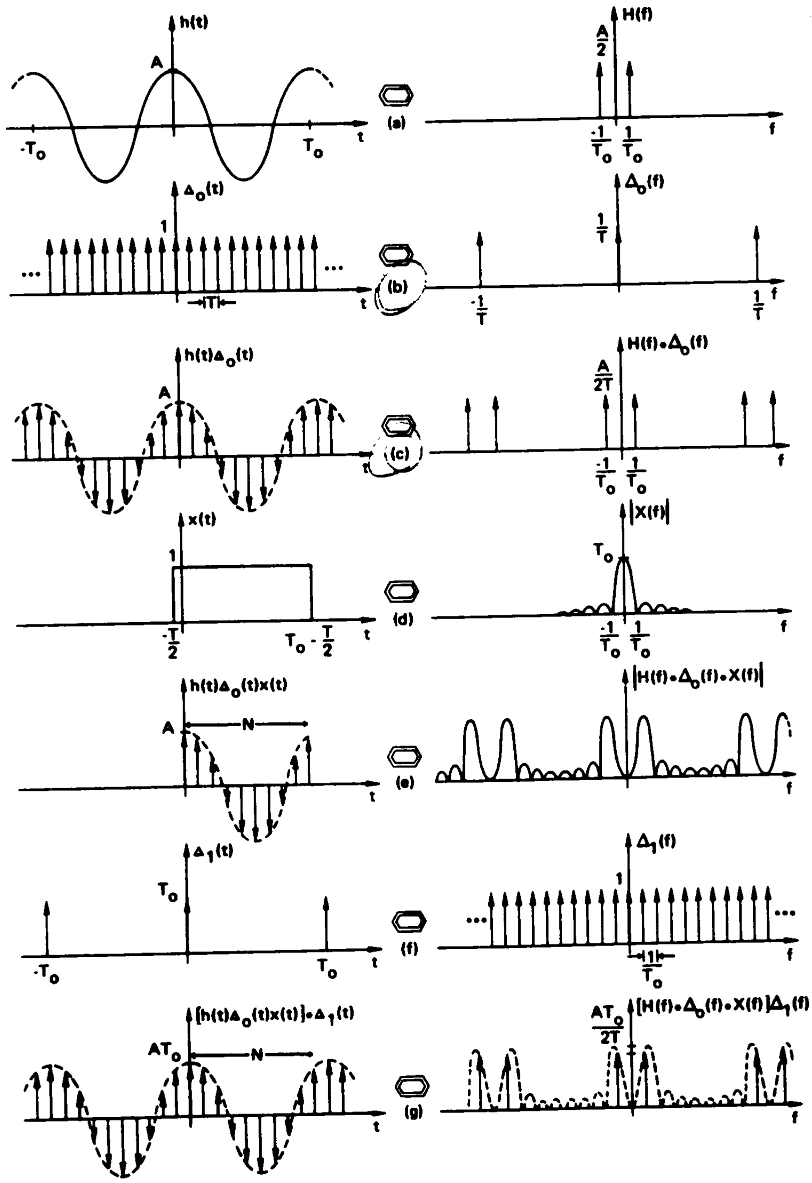


Figure 6.3 Discrete Fourier transform of a band-limited periodic waveform: the truncation interval is equal to one period.

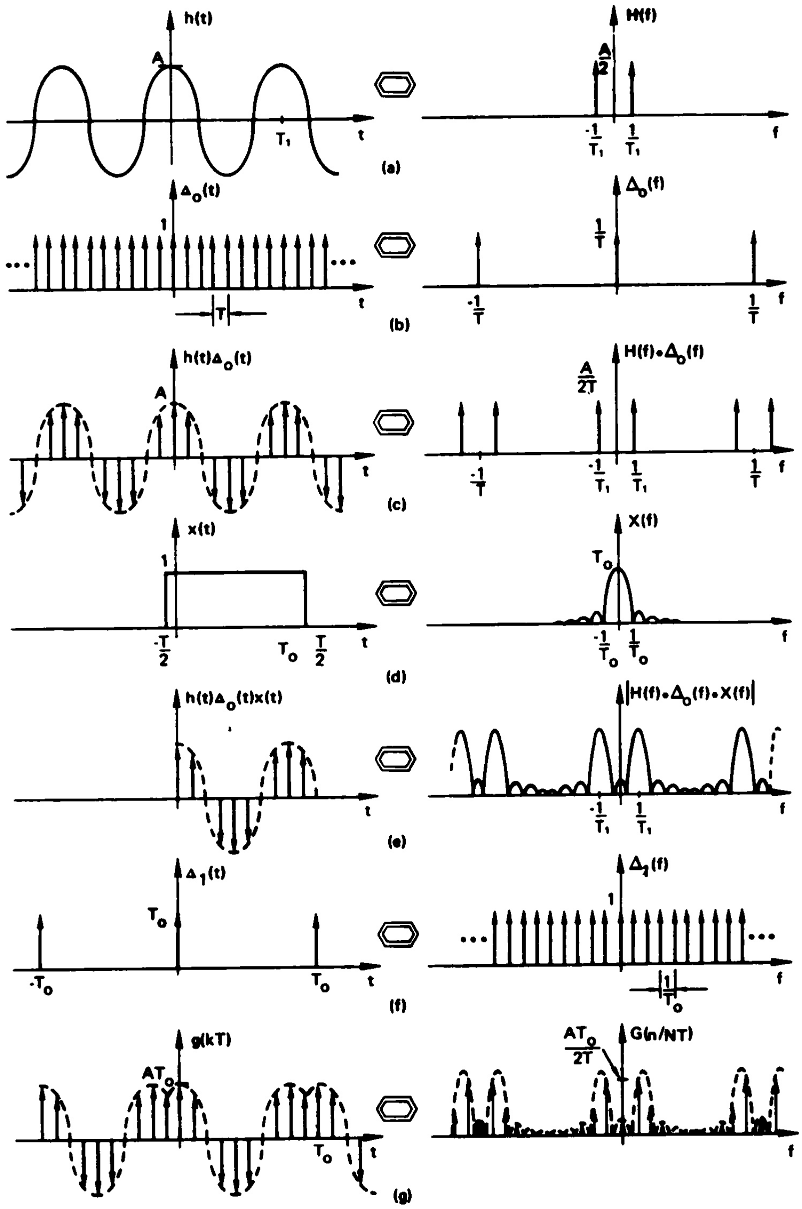


Figure 6.5 Discrete Fourier transform of a band-limited periodic waveform: the truncation interval is not equal to one period.

→ only N distinct values of $X(f = {}^{n}/N\Delta t) = \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j\, 2\pi k n/N}$

$\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \Longrightarrow$

$$\boxed{X(f = {}^{n}/N\Delta t) = \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j\, 2\pi n k/N} \qquad n = 0, \pm 1, \pm 2, \ldots}$$

Δt

inverse (without proof)

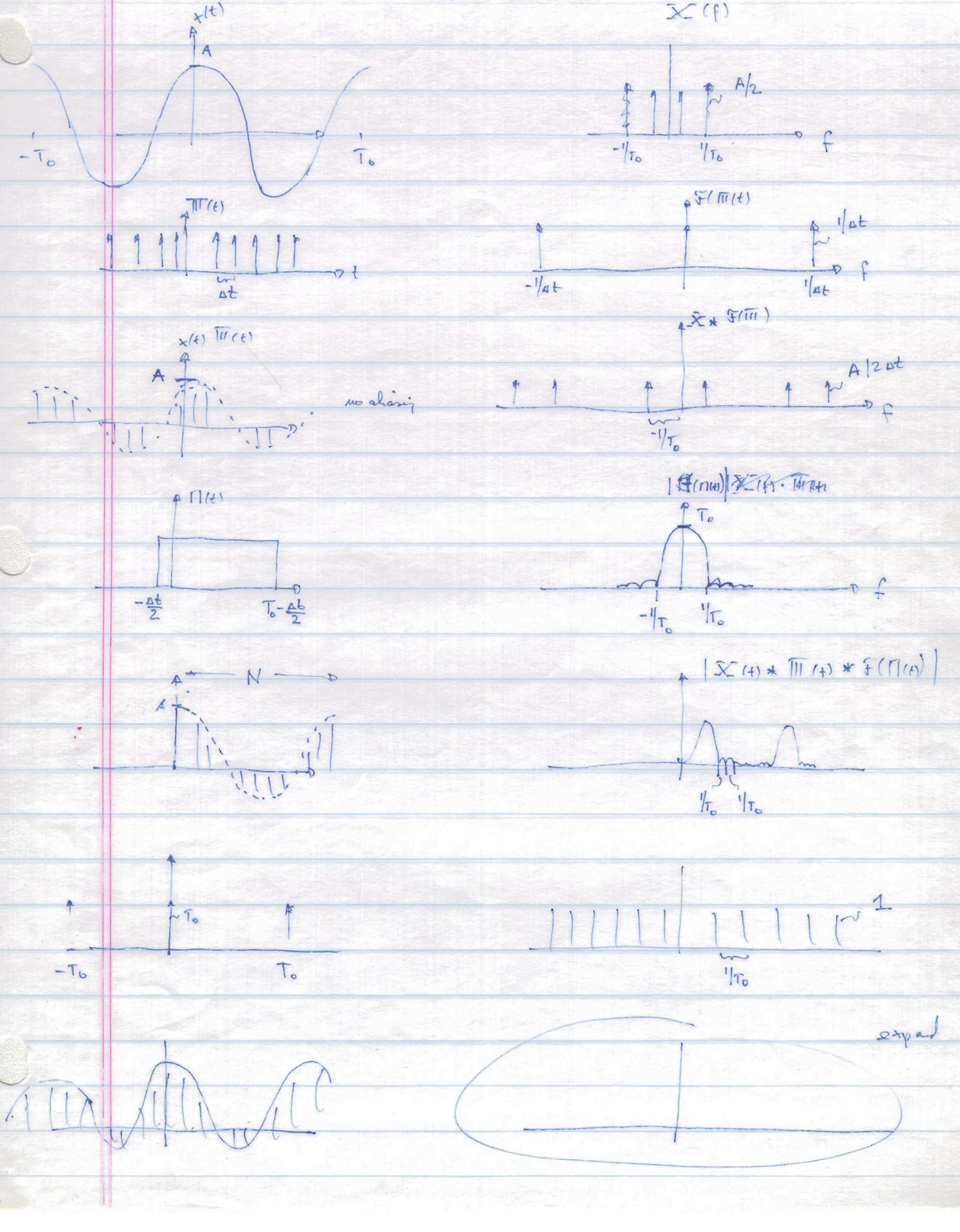
$\int_{-\infty}^{\infty} X\, e^{+j2\pi f t}\, df$

$$\boxed{\mathcal{F}^{-1}(X(f_n)) = x(t) = \frac{1}{N\Delta t} \sum_{n=0}^{N-1} X(f_n)\, e^{j\, 2\pi n k/N} \qquad k = 0, 1, 2, \ldots \quad f_n = n/N\Delta t}$$

Note that these formulae map one set of numbers (no dimensions) into another set of numbers (no dimensions either), i.e., it's exactly what we want → computers

class #5

Relation between discrete + continuous FT



Brigham, sections 10.1 + 10.2

CLASS #6

Discrete Fourier Transform

$$\underline{X}\left(f = n/N\Delta t\right) = \underline{X}_n = \Delta t \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j 2\pi \frac{nk}{N}} \qquad n = 0, 1, 2, \ldots N-1$$

set $\Delta t = 1$

lets assume $N = 4$, then we have

$$
\begin{aligned}
n=0 &: \quad X_0 = x_0 W^0 + x_1 W^0 + x_2 W^0 + x_3 W^0 \\
n=1 &: \quad X_2 = x_0 W^0 + x_1 W^1 + x_2 W^2 + x_3 W^3 \\
n=2 &: \quad X_2 = x_0 W^0 + x_1 W^2 + x_2 W^4 + x_3 W^6 \\
n=3 &: \quad X_3 = x_0 W^0 + x_1 W^3 + x_2 W^6 + x_3 W^9
\end{aligned}
$$

$\rightarrow N^2$ complex multiplications

where $W = e^{-j 2\pi / N}$

in matrix notation this is

$$\underset{\sim}{X}_n = \underset{\sim}{W}^{n \cdot k}\, \underset{\sim}{x}_k$$

or

$$\begin{pmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{pmatrix} = \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

note, however, that

$$W^{n \cdot k} = W^{n \cdot k \bmod (N)} \quad , \text{ i.e., } W^6 = W^2$$

i.e.,

$$W^6 = \left(e^{-j\,2\pi/N}\right)^6 = e^{-j\frac{2\pi}{4}\cdot 6} = e^{-j\,3\pi} = e^{-j\pi} = e^{-j\frac{2\pi}{4}\cdot 2}$$
$$= \left(e^{-j\,2\pi/N}\right)^2 = W^2$$

and $\quad W^0 = 1$

$\Rightarrow$
$$\begin{pmatrix} \tilde{X}_0 \\ \tilde{X}_1 \\ \tilde{X}_2 \\ \tilde{X}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 \\ 1 & W^2 & W^0 & W^2 \\ 1 & W^3 & W^2 & W^1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

now, check yourself that

$$\begin{pmatrix} \tilde{X}_0 \\ \tilde{X}_2 \\ \tilde{X}_1 \\ \tilde{X}_3 \end{pmatrix} = \begin{pmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^0 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{(y_0,\, y_1,\, y_2,\, y_3)}$$

This factorization is at the heart of the FFT

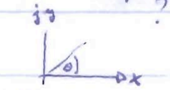
computation of (y_0, y_1, y_2, y_3) appears to require 4 complex multiplications, note, however that

$$W^0 = -W^2$$

$\rightarrow$ just 2 complex multiplications, but if $W^0 = 1$ and $W^2 = -1$ $\Rightarrow$ zero complex multiplications?

Yes, but at this point W^0 is not reduced to 1 for generality

$$W^2 = \left(e^{-j\,2\pi/N}\right)^2 = e^{-j\frac{2\pi}{N}\cdot 2} = e^{-j\pi} = \cos\pi + j\sin\pi = \cos\pi = -\cos 0$$
$$= -W^0$$



$e^{j\theta} = \cos\theta + i\sin\theta$

$e^{-j\theta} = \cos\theta - i\sin\theta$

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_2 \\ \tilde{x}_1 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

same as before, we just need 2 complex multiplications

$\Rightarrow$ total of 4 complex multiplications (+ 8 complex additions)

$N = 2^{\gamma}$ $\quad N \cdot \gamma/2 \qquad N \cdot \gamma$

# of multiplications $\frac{\text{discrete Fourier transform}}{\text{FFT}} = \frac{N^2}{N\gamma/2} = \frac{2N}{\gamma}$!

It all comes from the smart factorization of the $N \times N$ matrix into γ (where $N = 2^{\gamma}$) matrices that reduce the number of multiplications

Prior to the FFT you want to "taper" or "window" the data (in the time domain) in order to reduce leakage.