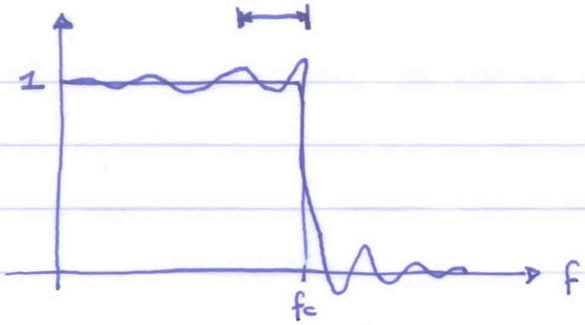
$\Delta f = \frac{2}{(2s+1)\Delta t}$ truncation due to finite window that is $T_W = (2s+1)\Delta t$ long, so

$$\frac{\Delta f}{2} = T_W$$



Δf is ~~the~~ the highest frequency that we can get for a finite window of samples/filterweights in the time domain: $\longrightarrow f_n = \frac{n}{T_W} = \frac{n \Delta f}{2}$

suppose we average in frequency over a ripple period (as we know its wavelength Δf), we then reduce the ripple

$$\Delta f = \frac{2 f_N}{N} = \frac{2}{N \cdot \Delta t}$$

$$N = (2s+1)$$

Define

$$\hat{H}_s(f) = \frac{1}{\Delta f} \int_{f-\frac{\Delta f}{2}}^{f+\frac{\Delta f}{2}} \hat{H}(\lambda)\, d\lambda$$

λ is a frequency

$$= \frac{1}{\Delta f} \int_{f-\frac{\Delta f}{2}}^{f+\frac{\Delta f}{2}} h_0 + 2 \sum_{n=1}^{s} h_n \cos(2\pi n \Delta t \lambda)\, d\lambda$$

$$= \frac{1}{\Delta f} \int_{f-\frac{\Delta f}{2}}^{f+\frac{\Delta f}{2}} h_0 \, d\lambda + \frac{1}{\Delta f} \int_{f-\frac{\Delta f}{2}}^{f+\frac{\Delta f}{2}} 2 \sum_{n=1}^{s} h_n \cos(2\pi n \Delta t \lambda)\, d\lambda$$

$$= h_0 + \frac{2}{\Delta f} \sum_{n=1}^{s} h_n \left. \frac{\sin(2\pi n \Delta t \lambda)}{2\pi n \Delta t} \right|_{\lambda = f - \frac{\Delta f}{2}}^{\lambda = f + \frac{\Delta f}{2}}$$

$$= h_0 + \frac{2}{\Delta f \cdot 2\pi} \sum_{n=1}^{s} \frac{h_n}{n \Delta t} \sin\left[2\pi n \Delta t (f + \Delta f/2)\right] - \sin\left[2\pi n \Delta t (f - \tfrac{\Delta f}{2})\right]$$

but $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$; $a = 2\pi n \Delta t f$, $b = 2\pi n \Delta t \Delta f/2$

$$\hat{H}_s(f) = h_0 + \frac{2}{\Delta f \cdot 2\pi} \sum_{n=1}^{s} \frac{h_n}{n \Delta t} \left(\cancel{\sin a \cos b} + \cos a \sin b - \cancel{\sin a \cos b} + \cos a \sin b \right)$$

$$= h_0 + \frac{2}{\Delta f \cdot 2\pi} \sum_{n=1}^{s} \frac{h_n}{n \Delta t} \cdot 2 \cos a \sin b$$

$$= h_0 + \frac{2}{\Delta f \cdot \pi} \sum_{n=1}^{s} \frac{h_n}{n \Delta t} \sin(2\pi n \Delta t \Delta f/2) \cos 2\pi n \Delta t f$$

$$= h_0 + \frac{2}{\pi \; 2/(2s+1)\Delta t} \sum_{n=1}^{s} \frac{h_n}{n \cdot \Delta t} \sin\left(2\pi n \cancel{\Delta t} \frac{\cancel{2}}{\cancel{2} \cdot (2s+1) \cancel{\Delta t}}\right) \cos(2\pi n \Delta t f)$$

$$= h_0 + \frac{(2s+1)}{\pi} \cancel{\Delta t} \sum_{n=1}^{s} \frac{h_n}{n \cancel{\Delta t}} \sin\left(\frac{2\pi n}{(2s+1)}\right) \cos(2\pi n \Delta t f)$$

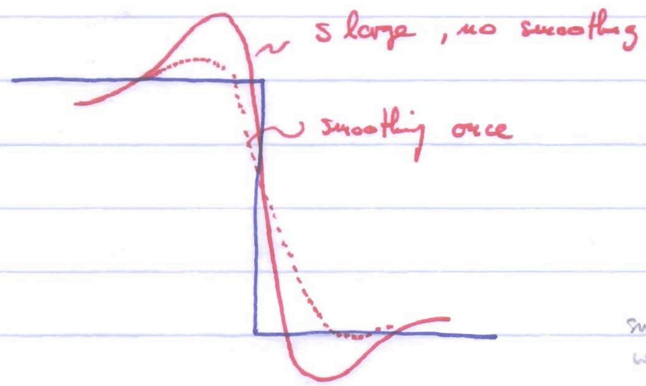
$$= h_0 + 2 \sum_{n=1}^{s} h_n \sin\left[\frac{2\pi n}{(2s+1)}\right] \cdot \frac{1}{2\pi n/(2s+1)} \cos 2\pi n \Delta t f$$

$$= h_0 + 2 \sum_{n=1}^{s} \underbrace{h_n \frac{\sin[2\pi n/(2s+1)]}{2\pi n/(2s+1)}}_{} \cos(2\pi n \Delta t f)$$

new filter weights,
do not depend upon f!

↳ the # of filterweights s and somewhat independent of f_c

the wavelength of the ripples $\propto$ s ; but the amplitude is reduced by the smoothing



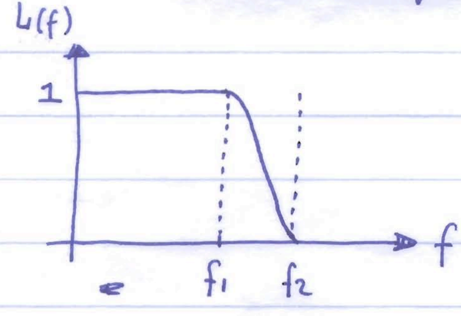
$$h'_n = h_n \cdot \sigma_n \qquad \sigma_n = \frac{\sin\left[\frac{2\pi n}{(2s+1)}\right]}{\frac{2\pi n}{(2s+1)}}$$

smoothed weights — unsmoothed weights

Thompson Filter (outline)

Thompson, R.O.R.Y. (1983). Low-pass filters to supress inertial and tidal frequencies. J. Phys. Oceanogr., 13, 1077-1083

design criteria :
- no ripples/ringing/leakage at relevant frequencies under any circumstances
- but recognize that there is no ideal filter



$$L(f, f_1, f_2) = \begin{cases} 1 & f < f_1 \\ \frac{1}{2}\left(1 + \cos\left[\frac{\pi(f - f_1)}{f_2 - f_1}\right]\right) & \\ 0 & f > f_2 \end{cases}$$

a very good filter results already if you do this in the frequency domain, i.e., multiply this fct. with your FFT of the data and apply the inverse FFT

see Forbes, A.M.G. (1988). Fourier transform filtering: A cautionary note. JGR, 93, 6958-6962

The Fourier transform of the real filterweights h_n ~~are~~ is

$$H(f) = h_0 + 2 \sum_{n=1}^{S} h_n \cos(2\pi n \Delta t f)$$

Next minimize the mean square ~~error~~ E between the real transform $H(f)$ and the ideal transform $L(f)$ in a least square sense, i.e.,

find h_n from $\frac{\partial E}{\partial h_n} = 0$ $\qquad E = \int_0^{f_{Nyquist}} \left[H(f) - L(f)\right]^2 df$

subject to

$$H(f=0) = 1$$
$$H(f=f_i) = h_0 + 2 \sum_{n=1}^{S} h_n \cos(2\pi n f_i \Delta t) = 0 \qquad i = 1, 2, \dots m$$

these are linear constraints of a linear least square problem ~~that~~ that can be solved through ~~a~~ Lagrange multiplies and ~~Eq~~ = a lot of algebra

→ see paper

main point here: filter design is up to you
anything goes
create your own filter to suit your purpose

~~sofo~~ class #7 finish

sofar we had only non-recursive filters of the form

$$y_n = \sum_{k=-S}^{S} c_k \, x_{n-k}$$

however, we can also find

$$y_n = \sum_{k=-N}^{N} c_k \, x_{n-k} + \sum_{k=1}^{M} d_k \, y_{n-k}$$

? ↓

$$H(f) = \frac{\sum_{l=0}^{N} c_l \, e^{-j 2\pi l f \Delta t}}{1 - \sum_{l=1}^{M} d_l \, e^{-j 2\pi l f \Delta t}}$$

advantages of a ~~non~~-recursive filter: fewer # of filter weights needed as the denominator may "blow up" → sharp cut off response with few M and N

Butterworth filter (Roberts + Roberts (1978) JGR)

disadvantages: - involved design
- can go "wild" → unstable
- introduce phase shifts, however, run forward then backward → recover phase