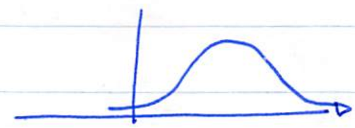
Some PDFs

(1) Normal or Gaussian Distribution $\quad N(\mu, \sigma^2)$

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$$



mean μ and variance σ^2

if $\mu = 0$ and $\sigma^2 = 1$ then standardized normal distribution $N(0,1)$

(2) Chi - Squared Distribution

Let $x_1, x_2, \dots x_n$ be the independent random variables with distribution $N(0,1)$ and define

$$\text{Chi}_n \equiv \chi_n^2 = x_1^2 + x_2^2 + \dots x_n^2$$

apps from Iowa University

$$p(\chi^2) = \frac{(\chi^2)^{n/2-1}\, e^{-\chi^2/2}}{2^{n/2}\,\Gamma(n/2)}$$

Γ is gamma fctn

χ_n^2 is called the chi-square distribution with n degrees of freedom

$$\Gamma(n) = (n-1)!$$
$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n-1)$$

"Mean" $\quad E[\chi_n^2] = \mu_{\chi^2} = n$

or

$\cancel{\Gamma(z) = \int_0^\infty t^{(z-1)} e^{-t}\, dt}$

"Variance" $\quad E[(\chi_n^2 - \mu_{\chi^2})^2] = \sigma_{\chi^2}^2 = 2n$

Come back to this after p.-68 $\quad \Gamma(z) = \int_0^\infty t^{(z-1)} e^{-t}\, dt$

As n becomes large the χ^2_n distribution approaches the normal distribution, specifically

$$n > 30 \quad \text{then} \quad \sqrt{2\chi^2_n} \approx N\left(\mu = \sqrt{2n-1}\,;\ \sigma^2 = 1\right)$$

HOW TO JUDGE QUALITY OF PARAMETER ESTIMATES ?

$$\mu_x = E[x] = \int_{-\infty}^{\infty} x\, p(x)\, dx \qquad \hat{\mu}_x = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\int_{-\infty}^{+\infty} x\, p(x)\, dx \neq \frac{1}{N}\sum_{i=1}^{N} x_i$$

criteria:

1. Estimate should be unbiased, i.e.,

$$E[\hat{\mu}_x] = \mu_x$$

mean square error of estimator shall be smaller than any other estimator

2. Estimate should be efficient, i.e.,

$$E[({}_1\hat{\mu}_x - \mu_x)^2] \leq E[({}_i\hat{\mu}_x - \mu_x)^2]$$

3. Estimate should be consistent, i.e.

$$\lim_{N \to \infty} \mathrm{Prob}[\,|\hat{\mu}_x - \mu_x| \geq \varepsilon\,] = 0$$

sufficient but no necessary condition for this is: $\lim_{N \to \infty} E[(\hat{\mu}_x - \mu_x)^2] = 0$

GOTO p. 55
that is: skip the exposition of these
9-27-2016

$$\overline{x} = \hat{\mu}_x = \frac{1}{N} \sum_{i=1}^{N} x_i$$ is unbiased and consistent estimate

$$S_b^2 = \hat{\sigma}_x^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$$ is both biased and inconsistent

but

$$S^2 = \sigma_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2$$ is unbiased and consistent

long and tedious algebraic proofs

If x is a normally distributed variable with a mean μ_x and a variance σ_x^2 then it can be shown (Brownlee, K. A., 1965) that

$$(N-1)\ \hat{\sigma}_x^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2 = \sigma_x^2\ \chi^2_{N-1}$$

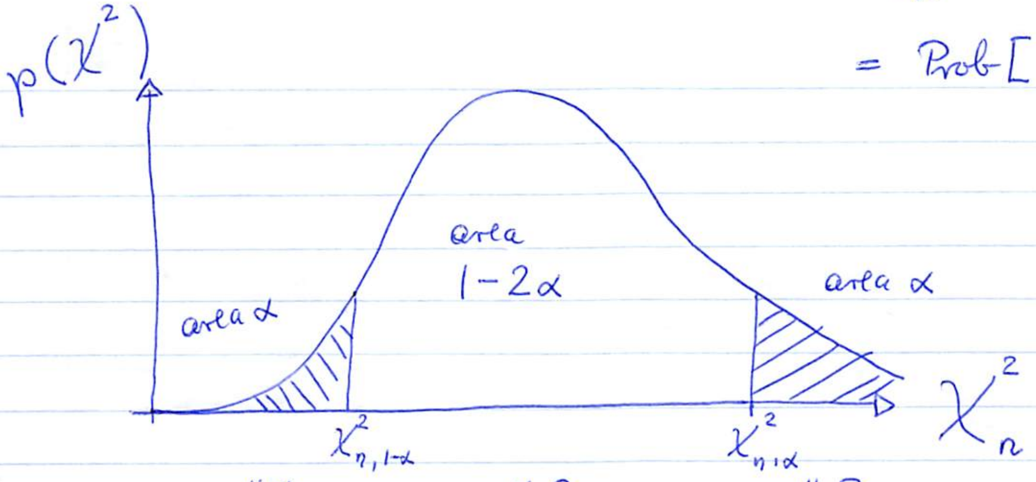
$$\boxed{\hat{\sigma}_x^2 = \frac{\sigma_x^2}{N-1} \chi^2_{N-1}}$$

use this to estimate uncertainty of variance estimates

estimated $\hat{\sigma}_x^2$ unknown true σ_x^2

$$P(x) = \int_{-\infty}^{x} x' \cdot p(x')\, dx' \qquad (52)$$

$$= \text{Prob}[\, x_k \le x \,] \qquad (53)$$



#1 #2 #3

$n = N-1$

discuss #3 before #2

#1 :

$$\alpha = \text{Prob}\left[\, \hat{\sigma}_x^2 \le \frac{\sigma_x^2}{n} \chi^2_{n,1-\alpha} \,\right]$$

#2 :

$$1-2\alpha = \text{Prob}\Big[\, \underbrace{\frac{\sigma_x^2}{n} \chi^2_{n,\alpha} > \hat{\sigma}_x^2}_{\#3} \underbrace{\ge \frac{\sigma_x^2}{n} \chi^2_{n,1-\alpha}}_{\#1} \,\Big]$$

$$= \text{Prob}\left[\, \frac{\chi^2_{n,\alpha}}{n \cdot \hat{\sigma}_x^2} > \frac{1}{\sigma_x^2} \ge \frac{\chi^2_{n,1-\alpha}}{n\, \hat{\sigma}_x^2} \,\right]$$

#3

$$\alpha = \text{Prob}\left[\, \hat{\sigma}_x^2 > \frac{\sigma_x^2}{n} \chi^2_{n,\alpha} \,\right]$$

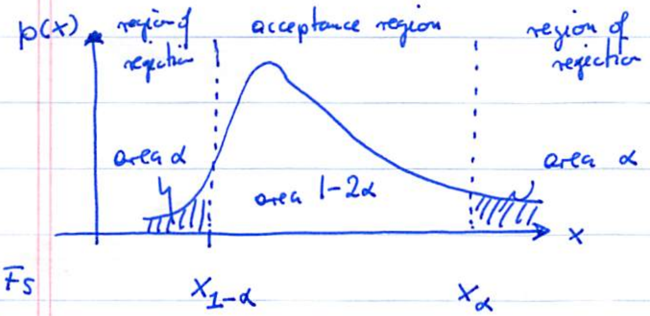
$\frac{n\hat{\sigma}^2}{\chi_{n,1-\alpha}}$

$$1-2\alpha = \text{Prob}\left[\, \frac{n\,\hat{\sigma}_x^2}{\chi^2_{n,\alpha}} < \sigma_x^2 \le \frac{n\,\hat{\sigma}_x^2}{\chi_{n,1-\alpha}} \,\right]$$

$\frac{n\hat{\sigma}_x^2}{\chi^2_{n,\alpha}}$

More generally

Hypothesis testing (Bendat + Piersol p. 88 – 91)



2α is the significance level

choose $\alpha = 0.025$ to get 95% confidence intervals (1 in 20 chance to be wrong)

note that assymetric PDFs (such as χ^2) give assymetric confidence levels, i.e.,

$$\text{Prob}\,[\,x \le x_{1-\alpha}\,] = \int_{-\infty}^{x_{1-\alpha}} p(x')\,dx' = \alpha$$

$$\text{Prob}\,[\,x > x_{\alpha}\,] = \int_{x_{\alpha}}^{\infty} p(x')\,dx' = \alpha$$

– statistical tests for

(a) Chi-square goodness of fit

(b) Normality of data

(c) Stationarity of data

see Bendat + Piersol (1986) chapter 4 for details and how to do these tests