Start class #9 in 1999 10/18/99

Auto-Spectral Analysis

stationary
random (from a normally distributed process)
ergodic
zero mean

Define Auto-covariance: $R_x(t,\tau) = E[x(t)\cdot x(t+\tau)]$

stationarity: $= E[x(t_1)\cdot x(t_2+\tau)] = \iint_{-\infty}^{\infty} x(t)\cdot x(t+\tau)\, p(x(t), x(t+\tau))\, dx(t)\, dx(t+\tau)$

ergodicity $= \lim_{T\to\infty} \underbrace{\frac{1}{T}\int_0^T x(t)\, x(t+\tau)\, dt}$

in practice we just have this, i.e., call it $_T R_x(\tau)$

$= R_x(\tau)$

$$_T R_x(\tau) = \frac{1}{T}\int_0^T x(t)\, x(t+\tau)\, dt$$

Properties

(1) $R_x(\tau) = R_x(-\tau)$ even fct.

(2) $|R_x(\tau=0)| \geq |R_x(\tau)|$

true variance lagged auto-covariance

(3) as the recordlength $T\to\infty$ the estimate of the auto-covariance

$$_T\hat{R}_x(\tau) \equiv \frac{1}{T}\int_0^T x(t)\, x(t+\tau)\, dt \qquad t+\tau < T$$

becomes asymptotically unbiased (Proof follows)

Proof of (3): assymptotically unbiased estimate

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$$E[{}_T\hat{R}(\tau)] = E\left[\frac{1}{T}\int_0^T x(t)\,x(t+\tau)\,dt\right] \qquad t+\tau < T$$

$$= \frac{1}{T}\int_0^T E[x(t)\,x(t+\tau)]\,dt$$

$$= \frac{1}{T}\int_0^{T-|\tau|} E[x(t)\,x(t+\tau)\,dt$$

? finite record length as $x(t)=0$ for $t>T$, $t<0$ ✓

$$= \frac{1}{T}\int_0^{T-|\tau|} \overset{E[\cdot]}{\cancel{{}_TR_x(\tau)}}\,dt = \frac{1}{T}\int_0^T E[\cdot]\,dt - \frac{1}{T}\int_0^{|\tau|} E[\cdot]\,dt$$

$$= {}_TR_x(\tau) - \frac{|\tau|}{T}\,\frac{1}{|\tau|}\int_0^{|\tau|} E[\,]\,dt$$

$$\boxed{= {}_TR_x(\tau)\,\left(1 - \tau/T\right)} \quad \rightarrow \text{ biased estimate !}$$

$$= {}_TR_x(\tau) - \frac{|\tau|}{T}\,{}_TR_x(\tau)$$

Definition: Auto-Spectrum $S_x(f)$

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau = \mathcal{F}(R_x(\tau))$$

$$= 2\int_0^{\infty} R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

because $R_x(\tau) = R_x(-\tau)$, $R_x(\tau)$ is even fctn.

and it also follows that

$$S_{xx}(-f) = S^{*}_{xx}(f) = S_{xx}(f)$$

Hence $S_{xx}(f)$ is real!

Hence

$$S_{x\cancel{x}}(f) = 2\int_0^\infty R_{x\cancel{x}}(\tau)\cos(2\pi f\tau)\,d\tau$$

and

$$\cancel{R_{x\cancel{x}}(f)}\ R_x(\tau) = 2\int_0^\infty S_x(f)\cos(2\pi f\tau)\,df$$

are Fourier Transform pairs if $\int_{-\infty}^{\infty} |R(\tau)|\,d\tau < \infty$

which in practice is always true because of finite record lengths.

for $\tau = 0$

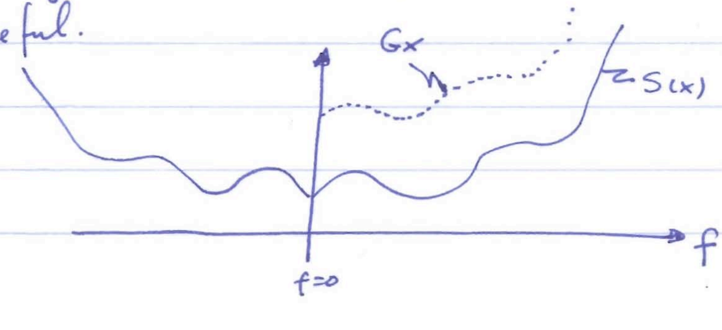
$$R_x(\tau=0) = \boxed{2\int_0^\infty S_x(f)\,df}$$

but it is also
(from its definition)

$$\underset{\text{true variance}}{R_x(\tau=0)} = \lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t)\,dt = \int_{-\infty}^{\infty} S_x(f)\,df$$

$S_x(f)$
PARTITION OF VARIANCE IN FREQUENCY SPACE

Hence we can interpret S_x as an energy (or variance) density function in the sense that the area under the curve $S_x(f)$ represents an energy or a variance. It is this physical (as opposed to the "messy" statistical) interpretation that makes the spectrum so useful.



$\cancel{G_x(f)=2}$

$$G_x(f) = \begin{cases} 2S_x & f \geq 0 \\ 0 & f < 0 \end{cases}$$

Spectral density ~~of~~ from a finite record : ${}_TS_x(f)$

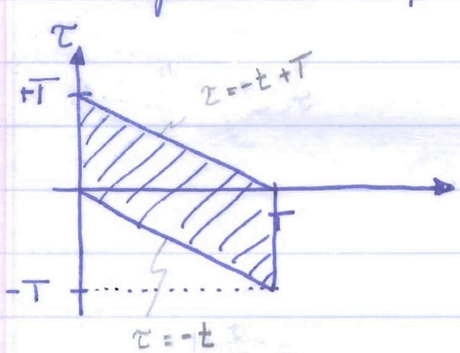
An estimate of that is

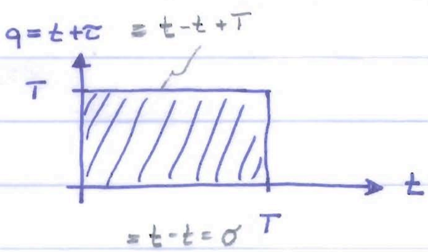
$${}_T\hat{S}_x(f) = \int_{-T}^{T} {}_T\hat{R}_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad (5:64)$$

$$= \int_{-T}^{T} \frac{1}{T}\int_{0}^{T} x(t)\, x(t+\tau)\, dt\; e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{1}{T}\int_{0}^{T} x(t) \int_{-T}^{T} x(t+\tau)\, e^{-j2\pi f\tau}\, d\tau\, dt$$

Now remember that ${}_T\hat{R}_x = 0$ for $t+|\tau| > T$, hence the actual integration in (τ, t) space is





Same "trick" use in proving the convolution theorem on p. 16

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change of variables $q = t+\tau$

$q = t+\tau$ or $\tau = q-t$

$dq = d\tau$

so $e^{-j2\pi f\tau}$

$= e^{-j2\pi f(q-t)}$

$= e^{-j2\pi f q} \cdot e^{+j2\pi f t}$

$${}_T\hat{S}_x(f) = \frac{1}{T}\int_{0}^{T} x(t)\, e^{+j2\pi f t} \int_{0}^{T} x(q)\, e^{-j2\pi f \cancel{\tau}\, q}\, \cancel{d\tau}\, dq\, dt$$

$${}_T\hat{S}_x(f) = \frac{1}{T}\left| \int_{0}^{T} x(t)\, e^{-j2\pi f t}\, dt \right|^2$$

or

$$\boxed{{}_T\hat{S}_x(f) = \frac{1}{T} X(f) \cdot X^*(f)}$$

where

$$X(f) = \int_0^T x(t)\, e^{-j 2\pi f t}\, dt$$

and

$$X^*(f) = \int_0^T x(t)\, e^{+j 2\pi f t}\, dt$$

is the complex conjugate of $X(f)$

end class #9

CLASS #10

1999
2009

SUMMARY

1. (a) stationary but not necessarily ergodic process $\{x_k(t)\}$ $k=1,2,\ldots$

$$R_x(\tau) = E[x_k(t)\,x_k(t+\tau)] = E[x_1 x_2] \quad \text{where } x_1 = x_k(t),\; x_2 = x_k(t+\tau)$$

→ missing step

$$= \iint_{-\infty}^{\infty} x_1 x_2\, p(x_1, x_2)\, dx_1\, dx_2$$

"ensemble mean"
"expected value"
"average over all possible outcomes"

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_k(t)\, x_k(t+\tau)$$

(b) ergodic process

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\, x(t+\tau)\, dt$$

2. $$E[{}_T\hat{R}_x(\tau)] = {}_TR_x(\tau)\left(1 - \frac{|\tau|}{T}\right)$$

3. $${}_T\hat{S}_x(f) = \int_{-T}^{T} {}_T\hat{R}_x(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{1}{T}\left|\int_0^T x(t)\, e^{-j2\pi f t}\, dt\right|^2$$

$$\boxed{{}_T\hat{S}_x(f) = \frac{1}{T}\, X(f)\cdot X^*(f)} \qquad X(f) = \mathcal{F}(x(t))$$