What is the expected value of the spectral density estimate for a finite record length?

$$E[{}_T\hat{S}_x(f)] = E\left[\int_{-T}^{T} {}_T\hat{R}_x(\tau)\, e^{-j2\pi f\tau}\, d\tau\right]$$

$$= \int_{-T}^{T} \underbrace{E[{}_T\hat{R}_x(\tau)]}_{\text{go over last class notes again (corrected?)}}\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-T}^{T} {}_T R_x(\tau)\left(1 - \frac{|\tau|}{T}\right) e^{-j2\pi f\tau}\, d\tau$$

see p.-56

$$= {}_T S_x(f) - \int_{-T}^{T} |\tau|/T \; e^{-j2\pi f\tau}\, d\tau$$

$$\neq {}_T S_x(f) \quad \rightarrow \text{biased estimate, but}$$

$$\lim_{T\to\infty} E[{}_T\hat{S}_x(f)] = \int_{-\infty}^{\infty} {}_T R_x(\tau)\, e^{-2\pi j f\tau}\, d\tau = {}_T S_x(f)$$

hand waving

assymptotically unbiased estimate

— " — consistent "

$$ {}_T\hat{S}_x(f) = \frac{1}{T} X(f) \cdot X^*(f) $$

The discrete version of the estimated power spectral density fctn. we get from the FFT, i.e.

$$ \boxed{ X\left(f_n = {}^{n}/_{N\Delta t}\right) = \Delta t \sum_{\ell=0}^{N-1} \underbrace{x(\ell\cdot\Delta t)}_{x_\ell} e^{-j 2\pi \frac{n\cdot\ell}{N}} \qquad n = 0, 1, \ldots N-1 \qquad f_n = \frac{n}{T} \quad T = N\cdot\Delta t } $$

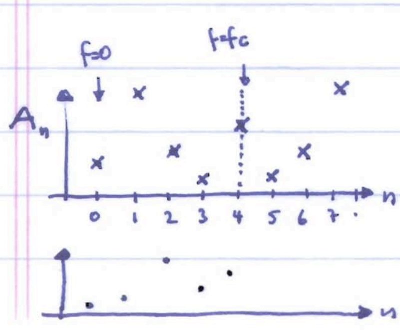
Note, however, that for $n = N/2$ $\quad f_{\frac{N}{2}} = {}^{N}/_{2N\Delta t} = {}^{1}/_{2\Delta t}$

is the Nyquist frequency, i.e., there are only $N/2$ independent values of X.

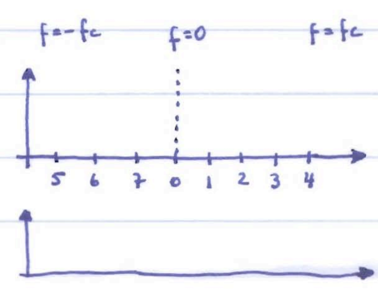
Another way to see this is by considering

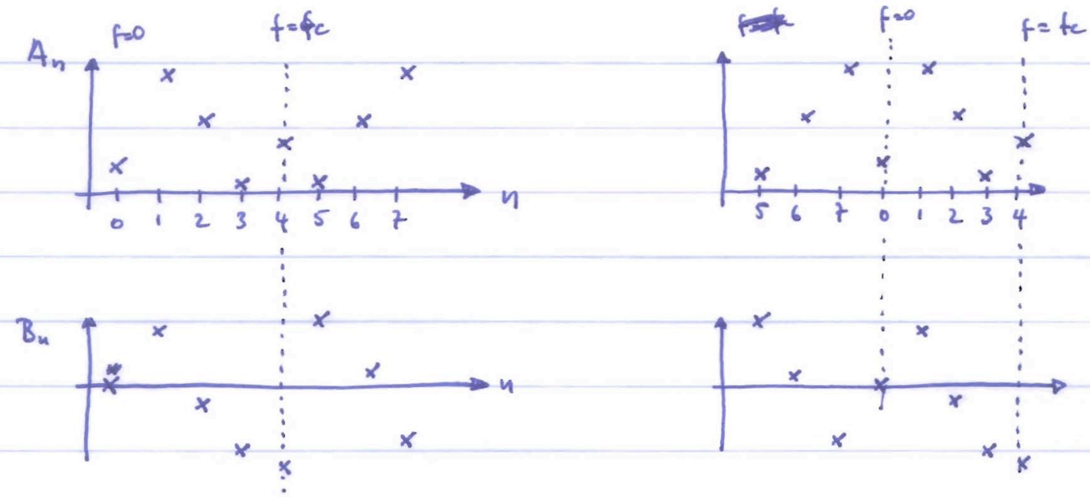
$$ X\left(f = {}^{N-n}/_{N\Delta t}\right) = \Delta t \sum_{\ell=0}^{N-1} x(\ell\Delta t)\, e^{-j 2\pi \frac{(N-n)\ell}{N}} $$

$$ = \Delta t \sum_{\ell=0}^{N-1} x(\ell\Delta t)\, e^{+j 2\pi \frac{n\ell}{N}} \cdot \underbrace{e^{-j 2\pi \ell}}_{\equiv 1} $$



$$ = X^*\left(f = {}^{n}/_{N\Delta t}\right) $$




f=0
f=fc
A_n
0 1 2 3 4 5 6 7
n
B_n
n
f=0
f=fc
5 6 7 0 1 2 3 4

Joe 9am Wed
Claire 9am Thr
Arun 2pm Wed
Zhao 11pm Thu
Dal 11pm Thr

$$X(f = n/N\Delta t) = \Delta t \sum_{\ell=0}^{N-1} x(\ell \Delta t)\, e^{-j 2\pi \frac{n\ell}{N}} \qquad n = 0, 1, 2, \ldots N-1$$

$$= \Delta t \left(A_n - j\, B_n\right) \qquad A_n, B_n \text{ real}$$

$$X^*(f = n/N\Delta t) = \Delta t \left(A_n + j\, B_n\right)$$

Then

$$\hat{{}_T S_x}(f = n/N\Delta t) \equiv \hat{P}_n = \frac{1}{N\Delta t} X_n X_n^* \qquad n = 0, 1, \ldots N-1$$

two sided spectra

$$\boxed{\cancel{= \frac{1}{N\Delta t} X_n \cdot X_{N-n}}} \qquad n = 0, 1, \ldots N/2$$

one sided spectra

$(S_x(f) = S_x(-f))$

$$= \frac{\Delta t^2}{N\Delta t}\left(A_n^2 + B_n^2\right) = \frac{(A_n^2 + B_n^2)}{\cancel{\Delta t} \cdot N / \Delta t}$$

amplitude^2 per unit frequency

where

$$A_n = \cancel{\Delta t} \sum_{\ell=0}^{N-1} x_\ell \cos(2\pi \ell n / N)$$

$$B_n = \cancel{\Delta t} \sum_{\ell=0}^{N-1} x_\ell \sin(2\pi \ell \cdot n / N)$$

This looks like the amplitudes for a Fourier series !!!

~~Recall that~~

$$\cancel{\chi_n^2 = x_1^2 + x_2^2 + \ldots + x_n^2}$$

~~has~~ $\cancel{p(\chi^2) = (\chi^2)^{n/2-1} e^{-\chi^2/2} / 2^{n/2}\, \Gamma(n/2)}$

~~if x_i are independent random variables with $N(0;1)$~~

What are the expected values of these amplitudes squared?

will get us error estimates

$$E[A_n^2] = E\left[\sum_{k=0}^{N-1} x_k \cos(2\pi k n/N) \cdot \sum_{k=0}^{N-1} x_k \cos(2\pi k n/N) \right]$$

$$= E\left[\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} x_k x_l \cos(2\pi k n/N) \cos(2\pi l n/N) \right]$$

$\cos x \cdot \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$

$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} E[x_k x_l] \cos(2\pi k n/N) \cos(2\pi l n/N)$$

$$= \frac{1}{2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} E[x_k x_l] \left\{ \underbrace{\cos(2\pi (k-l) n/N)}_{1 \text{ for } k=l} + \cos(2\pi (k+l) n/N) \right\}$$

x_k and x_l are independent random variables, i.e., for $\boxed{k \neq l}$!

$$E[x_k x_l] = \iint_{-\infty}^{\infty} x_k x_l \, p(x_k, x_l) \, dx_k \, dx_l$$

$$= \iint_{-\infty}^{\infty} x_k x_l \, p(x_k) \, p(x_l) \, dx_k \, dx_l$$

$$= \int_{-\infty}^{\infty} x_k \, p(x_k) \, dx_k \cdot \int_{-\infty}^{\infty} x_l \, p(x_l) \, dx_l$$

$$= E[x_k] \cdot E[x_l]$$

$$= \mu_{x_k} \cdot \mu_{x_l} = 0$$

for $k \neq l$

mean has been removed, i.e. x have ~~dist~~ normal distributions $N(0, \sigma^2)$

Remove mean prior to spectral analyses !!!

For $k = l$:

$$E[A_n^2] = \frac{\cancel{\ }^2}{2} \sum_{l=0}^{N-1} E[x_k^2]\left\{\cos\left(2\pi\, \overset{k}{\vee} 2n/N\right) + 1\right\}$$

$$= \frac{\cancel{\ }^2}{2}\, \sigma^2 \sum_{l=0}^{N-1} \cos\left(2\pi\, n \overset{k}{\vee}/(N/2)\right) + 1$$

$$= \frac{\cancel{\ }^2}{2}\, \sigma^2 \left\{ N + \underbrace{\sum_{l=0}^{N-1} \cos\left(4\pi\, n \overset{k}{\vee}/N\right)}_{0 \text{ for } n \neq \sigma} \right\}$$

$$= \frac{\cancel{\ }^2}{\cancel{2}} \begin{cases} \sigma^2 \dfrac{N}{2} & \text{for } n \neq \sigma \\ \dfrac{2\sigma^2 N}{2} & \text{for } n = \sigma \end{cases}$$

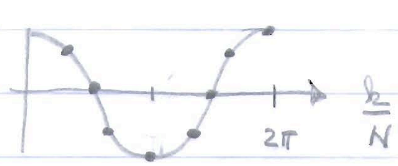
Hence

$$E[A_n^2] = \frac{N\sigma^2}{2} \qquad \text{for } n \neq 0$$

$$E[A_n^2] = N\sigma \qquad \text{for } n = 0$$

$$\left.\begin{aligned} E[B_n^2] &= \sigma && \text{for } n = 0 \\ E[B_n^2] &= \frac{N\sigma^2}{2} && \text{for } n \neq 0 \end{aligned}\right\} \text{similar arguments as for cosine}$$

$$\sum_{k=0}^{N-1} \cos\frac{2\pi\, n\cdot k}{N} = \sum_{k=1}^{N} \cos\frac{2\pi\, n\cdot k}{N} = \sum_{k=1}^{N} \cos 2\pi \frac{k}{N} =$$



$$= \cos\left(2\pi \frac{1}{N}\right) + \cos\left(2\pi \frac{2}{N}\right) + \dots + \cos\left(2\pi \frac{N}{N}\right)$$

Recap

$$_T\hat{S}_x(f) = \frac{1}{T} X \cdot X^*$$

$$_T\hat{S}_x(f) \equiv \hat{P}_n = \frac{\Delta t^2}{N \Delta t} \left(A_n^2 + B_n^2\right) \qquad n = 0, 1, \dots N-1$$

where $$A_n = \sum_{\ell=0}^{N-1} x_\ell \cos(2\pi \ell n / N)$$

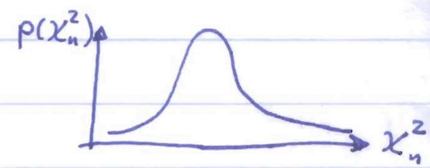
$$n = 0, 1, \dots N/2$$

$$B_n = \sum_{\ell=0}^{N-1} x_\ell \sin(2\pi \ell n / N)$$

The data x_ℓ were independent random variables drawn from a Gaussian or Normal distribution with mean $\mu_x = 0$ and variance σ_x^2, i.e., $N(0; \sigma_x^2)$

The (A_n, B_n) are linear fctn of the x_ℓ and are thus also normally distributed, however, their variance $E[A_n^2]$ and $E[B_n^2]$ is $N\sigma_x^2/2$ for $n \neq 0$. (proof p.65-66)

The $\hat{P}_n$ are the sum of squares, i.e., $\hat{P}_n \propto (A_n^2 + B_n^2)$ and thus are NOT normally BUT chi-square distributed.



Recall that

$$\chi_n^2 = x_1^2 + x_2^+ + \dots + x_n^2$$

has

$$p(\chi_n^2) = (\chi^2)^{n/2-1} e^{-\chi^2/2} \Big/ 2^{n/2} \Gamma_{(n/2)}$$

for x_i that are $N(0,1)$

CLASS #11

Now

$$\hat{P}_n = \frac{\Delta t^2}{N \Delta t} (A_n^2 + B_n^2)$$

for x_i from $N(\mu_x = 0; \sigma_x^2 = 1)$

$\chi_n^2 = x_1^2 + x_2^2 + \dots x_n^2$

see p. 49

and

$$\frac{\hat{P}_n}{\Delta t\, \sigma_x^2/2} = \frac{A_n^2}{N \sigma_x^2/2} + \frac{B_n^2}{N \sigma_x^2/2} = \chi_2^2$$

because $E[A_n^2] = N\sigma_x^2/2$

$E\left[\frac{A_n^2}{N\sigma_x^2/2}\right] = 1$!

Hence $\frac{A_n}{\sigma_x\sqrt{N/2}}$ are $N(0,1)$

end of #10

Start class #11

$$E[\chi_n^2] = n \qquad \text{(page 49)}$$

$$E[2\hat{P}_n/(\Delta t\, \sigma_x^2)] = E[\chi_2^2] = 2$$

$$E[\hat{P}_n/(\Delta t\, \sigma_x^2)] = 1$$

$$E[\hat{P}_n/\Delta t] = \sigma_x^2 = P_n/\Delta t$$

this is where I define P_n as an unbiased variable

Note that the expected value of $\hat{P}_n$ does not depend on N !

Hence $\lim_{N \to \infty} E[(\hat{P}_n - P_n)^2] = E[(\hat{P}_n - P_n)^2] = 2n \neq 0$

check notes when I talked about χ_n^2 p. 49

→ inconsistent estimate $\hat{P}_n$

What does this mean? Increasing the number of data points N does NOT make a better estimate $\hat{P}_n$