Why?

Given a time series with N points, you'll get $N/2+1$ independent $\hat{P}_n$

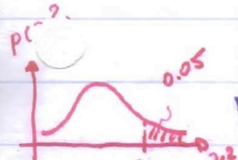
Given a time series with 2N points, you'll get $2N/2+1 = N+1$ independent $\hat{P}_n$

→ Increasing the length of the time series increases the frequency resolution ($\Delta f = 1/T = 1/N\Delta t$) and nothing else

How good are estimates $\hat{P}_n$?

$$\frac{\hat{P}_n}{\Delta t\,\sigma_x^2/2} = \frac{2\hat{P}_n}{P_n} = \chi_2^2 \qquad P_n = \Delta t\,\sigma_x^2 \qquad E[\hat{P}_n] = P_n$$

from p.68 from p.68



$$\text{Prob}[\chi_2^2 > 6] = 0.05 = \text{Prob}\left[\frac{2\hat{P}_n}{P_n} > 6\right] = \text{Prob}[\hat{P}_n > 3P_n] = \text{Prob}[\hat{P}_n/3 > P_n]$$

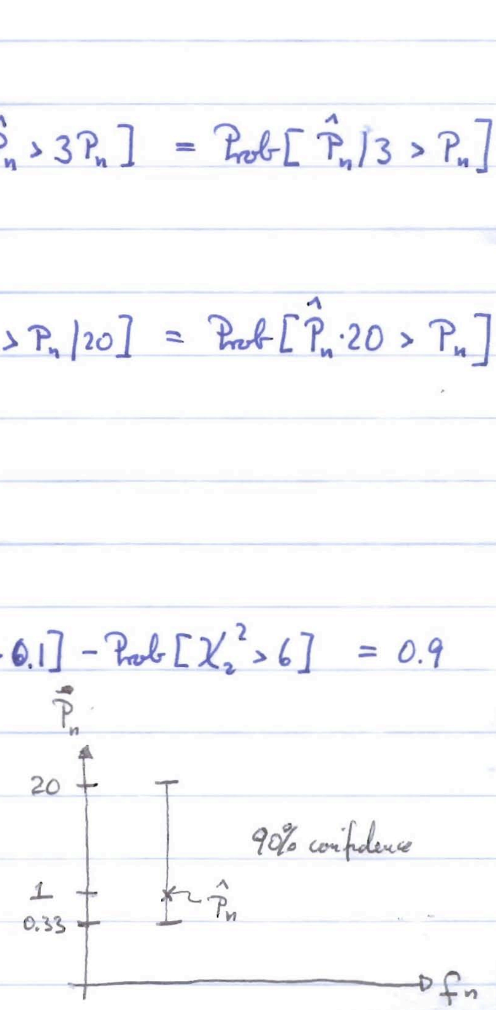
$$\text{Prob}[\chi_2^2 > 0.1] = 0.95 = \text{Prob}\left[\frac{2\hat{P}_n}{P_n} > 0.1\right] = \text{Prob}[\hat{P}_n > P_n/20] = \text{Prob}[\hat{P}_n \cdot 20 > P_n]$$

↓

$$\text{Prob}[\chi_2^2 \overset{<}{>} 6 \text{ and } \chi_2^2 < 0.1] = \text{Prob}[\chi_2^2 > 0.1] - \text{Prob}[\chi_2^2 > 6] = 0.9$$

↓

$$\text{Prob}\left[\frac{\hat{P}_n}{3} \le P_n \le 20\hat{P}_n\right] = 0.9$$



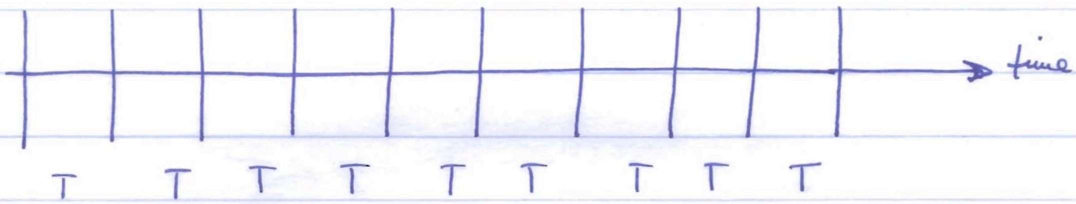
There is a 90% chance that the ~~estimate~~ true variance $\bar{P}_n$ lies between a third and 20 times the estimated variance $\hat{P}_n$.

→ This "confidence" interval is rediculous large.

→ Increase the d.o.f. somehow

How?

Do the FFT on pieces of the time series and average the estimates from each piece together.



Each of the "m" segment consists of N points $\quad \overline{\hat{P}_n} = \frac{1}{m} \sum_{r=1}^{m} |X_{n r}|^2$

For 2 segments, i.e., $m=2$

$$\overline{\hat{P}_n} = \frac{1}{2}\left(\hat{P}_{n,1} + \hat{P}_{n,2}\right)$$

$$= \frac{\Delta t^2}{N \Delta t} \cdot \frac{1}{2}\left(A_{n,1}^2 + A_{n,2}^2 + B_{n,1}^2 + B_{n,2}^2\right)$$

$$\cancel{4\,\frac{\overline{\hat{P}_n}}{P_n} = \chi_4^2} \qquad \cancel{E[\overline{\hat{P}}}$$

for x_i from $N(0,1)$
$\chi_n^2 = x_1^2 + x_2^2 + \dots + x_n^2$

and

$$\frac{\overline{\hat{P}_n}}{\Delta t\, \sigma_x^2/4} = \frac{A_{n,1}^2}{N\sigma_x^2/2} + \frac{A_{n,2}^2}{N\sigma_x^2/2} + \frac{B_{n,1}^2}{N\sigma_x^2/2} + \frac{B_{n,2}^2}{N\sigma_x^2/2} = \chi_4^2$$

because $E[A_{n,i}^2] = N\sigma_x^2/2$

and thus $E\left[\frac{A_{n,i}^2}{N\sigma_x^2/2}\right] = 1$

hence $\frac{A_{n,i}}{\sigma_x\sqrt{N/2}}$ is $N(0,1)$

$\Rightarrow$

$$E[4\overline{\hat{P}}/\Delta t\, \sigma_x^2] = E[\chi_4^2] = 4$$

$\Rightarrow$ $E[\overline{\hat{P}}/\Delta t] = \sigma_x^2 = P_n/\Delta t$ or $P_n = \Delta t\, \sigma_x^2$

Extend this procedure to a time series consisting of m segments each $N \cdot \Delta t$ long

$\Rightarrow$

$$\frac{2m\,\overline{\hat{P}_n}}{P_n} = \chi_{2m}^2$$

where $P_n/\Delta t = \sigma_x^2$
and $2m$ are the degrees of freedom

$E\left[\left(\frac{2m\,\overline{\hat{P}_n}}{P_n}\right)^2\right] = 2m$

$\Rightarrow$

$\frac{4m^2}{P_n^2} E[\overline{\hat{P}}^2] = 2m$

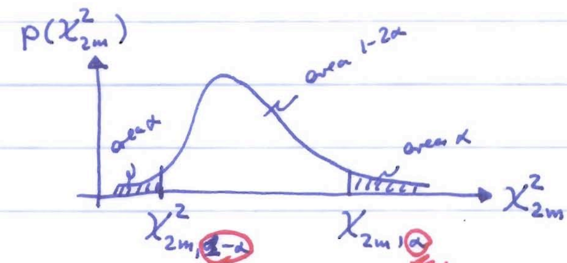
$\Rightarrow$

$E[\overline{\hat{P}}^2] = \frac{1}{2m} P_n$

$\Rightarrow$

$\lim_{m\to\infty} E[\overline{\hat{P}}^2] = 0$

$\Rightarrow$ consistent estimate!



say which direction the "area" is refered to

$$\mathrm{Prob}[\chi_{2m}^2 > \chi_{2m,\alpha}^2] = \alpha$$

$$\mathrm{Prob}[\chi_{2m}^2 > \chi_{2m,1-\alpha}^2] = 1-\alpha$$

$$\mathrm{Prob}[\chi_{2m,\alpha}^2 \geq \chi_{2m}^2 > \chi_{2m,1-\alpha}] = \mathrm{Prob}[\chi_{2m}^2 > \chi_{2m,1-\alpha}^2] - \mathrm{Prob}[\chi_{2m}^2 > \chi_{2m,\alpha}^2]$$

$$= 1 - 2\alpha$$