$$\lim_{m\to\infty} E[(\overline{\hat{P}}_n - P_n)^2] \overset{!}{=} 0$$ to prove consistency

$$E[\chi_n^2] = \mu_{\chi_n^2} = n$$

$$VAR[\chi_n^2] = E[(\chi_n^2 - \mu_{\chi_n^2})^2] = 2n$$

$$= E[(\chi_n^2)^2] - 2E[\chi_n^2 \mu_{\chi_n^2}] + \mu_{\chi_n^2}^2$$

$$= E[(\chi_n^2)^2] - \mu_{\chi_n^2}^2$$

$$= E[(\chi_n^2)^2] - n^2$$

$$\chi_{2m}^2 = 2m \ \overline{\hat{P}}_n / P_n$$

$\Downarrow$

$$VAR[\chi_{2m}^2] = E\left[\left(\frac{2m\overline{\hat{P}}_n}{P_n}\right)^2\right] - 4m^2 = 4m$$

$$\frac{4m^2}{P_n^2} E[\overline{\hat{P}}_n^2] - 4m^2 = 4m$$

$\Downarrow$

$$\frac{4m^2}{P_n^2} E[\overline{\hat{P}}_n^2] = 4m(1+m)$$

$\Downarrow$ $$E[\overline{\hat{P}}_n^2] = \frac{4m(1+m)}{4m^2} P_n^2$$

need
$$E[(\overline{\hat{P}}_n - P_n)^2] = E[\overline{\hat{P}}_n^2 - P_n^2]$$
note that $E[\overline{\hat{P}}_n] = P_n$
then show
$$E[\overline{\hat{P}}_n^2] - 2P_n E[\overline{\hat{P}}_n] + E[P_n^2]$$
$$= E[\overline{\hat{P}}^2] - 2P_n^2 + P_n^2$$
$$= E[\overline{\hat{P}}^2 - P_n^2]$$

$\Downarrow$ $$E[\overline{\hat{P}}_n^2] = \frac{1}{m} P_n^2 + P_n^2$$

$\Downarrow$ $$\lim_{m\to\infty} E[\overline{\hat{P}}_n^2] = \lim_{m\to\infty} \frac{1}{m} P_n^2 + P_n^2$$

$\Downarrow$ $$\lim E[\overline{\hat{P}}^2 - P_n^2] = \lim_{m\to\infty} \frac{1}{m} P_n^2 = 0$$

$$1-2\alpha = \text{Prob}\left[\chi^2_{2m,\alpha} \geq \chi^2_{2m} > \chi_{2m,1-\alpha}\right] \cancel{= 1-2\alpha}$$

or with $\chi^2_{2m} = 2m\ \bar{\hat{P}}_n / P_n$

$$1-2\alpha = \text{Prob}\left[\chi^2_{2m,\alpha} \geq 2m\frac{\bar{\hat{P}}_n}{P_n} > \chi^2_{2m,1-\alpha}\right]$$

$$= \text{Prob}\left[\frac{1}{\chi^2_{2m,\alpha}} \leq \frac{P_n}{2m\bar{\hat{P}}_n} < \frac{1}{\chi^2_{2m,1-\alpha}}\right]$$

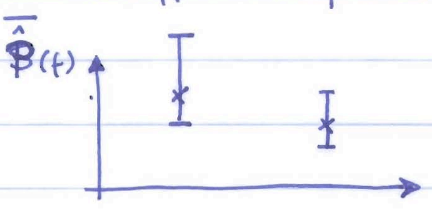
$$= \text{Prob}\left[\frac{2m\bar{\hat{P}}_n}{\chi^2_{2m,\alpha}} \leq P_n < \frac{2m\bar{\hat{P}}_n}{\chi^2_{2m,1-\alpha}}\right]$$ Bendat + Piersol (1986) p. 286 eq. 8.159

@ 95%; $m=10$: $0.95 = \text{Prob}\left[0.59\,\bar{\hat{P}}_n \leq P_n < 2.09\,\bar{\hat{P}}_n\right]$

$\log 0.59\,\bar{\hat{P}}_n \leq \log \hat{P}_n < \log 2.09\,\bar{\hat{P}}_n$

$\log 0.59 + \log \bar{\hat{P}}_n < \log \hat{P}_n \leq \log 2.09 + \log \bar{\hat{P}}_n$

The confidence intervals depend on the value of the estimates $\bar{\hat{P}}_n$, i.e., for each estimate I will have a different confidence interval.



~~or $\underset{m=10}{0.95} = \text{Prob}\left[0.59 \leq \underbrace{\log P_n}_{-\log \hat{P}_n} < \log 2.09\right]$~~

or $\underset{m=10}{0.95} = \text{Prob}\left[\log 0.59 \leq \underbrace{\log P_n - \log \hat{P}_n} < \log 2.09\right]$

$-0.23 \leq \underbrace{\text{uncertainty}}_{\log(P_n/\bar{\hat{P}}_n)} < 0.32$

On a log scale, however, the confidence interval becomes

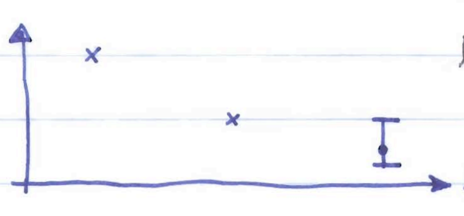
if $\bar{P} = 10$ $\log 10 = 1$

$1 - 0.23 = 0.77 \leq \underset{1}{\log P_n} \leq 1.32 = 1 + 0.32$

$$1-2\alpha = \text{Prob}\left[\log \bar{\hat{P}}_n + \log \frac{2m}{\chi_{2m,\alpha}} \leq \log P_n \leq \log \bar{\hat{P}}_n + \log \frac{2m}{\chi_{2m,1-\alpha}}\right]$$

ex $\alpha = 2.5\%$, $m = 10$: $0.95 =$ $\log\left(\frac{20}{34.17}\right)$ ... $\log\left(\frac{20}{9.59}\right)$

$\log(0.59) = -0.23$ ... $\log(2.09) = 0.32$



Now you can move the single "error" bar over each estimate.

Computational steps to estimate the distribution of variance in the frequency domain
(power spectral density function)

1. Divide the available data record for x_n , $n = 0, 1, 2, \ldots N \cdot m$ into m blocks each consisting of N data values

2. If needed to surpress sidelobe leakage due to finite record length, taper the data ~~of~~ each block x_n , $n = 0, 1, 2 \ldots N-1$ by the Hanning (or some other) taper/window

3. Compute the N-point FFT for each block of data giving $X(f_\ell)$, $\ell = 0, 1, \ldots N-1$

4. Adjust the scale factor of $X(f_\ell)$ for the loss of "variance" due to tapering (for Hanning taper, multiply by $\sqrt{8/3}$)

5. Compute the auto-spectral density estimate from m blocks of data as

any scale factors?

$$\overline{\hat{P}}(f_\ell) = \frac{1}{m} \sum_{n=1}^{m} X_n(f_\ell) \cdot X_n^*(f_\ell)$$

6. Compute confidence limits using the χ^2_{2m} distribution with 2m degrees of freedom for a specified confidence level (usually 95%).

(7.) Sometimes it is useful to present results in "variance preserving" form, i.e., multiply each estimate $\overline{\hat{P}}(f_\ell)$ by its frequency interval, i.e.,

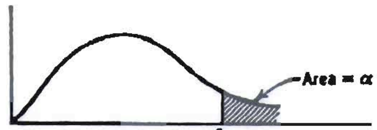
end of class #11

$$\overline{\hat{V}}(f_\ell) = \frac{\cancel{f_\ell} \cdot \overline{\hat{P}}(f_\ell)}{T}$$

Table A.3

Percentage Points of Chi-Square Distribution

Value of $\chi^2_{n;\alpha}$ such that $\text{Prob}[\chi^2_n > \chi^2_{n;\alpha}] = \alpha$



					α					
n	0.995	0.990	0.975	0.950	0.900	0.10	0.05	0.025	0.010	0.005
1	0.000039	0.00016	0.00098	0.0039	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.60
3	0.0717	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.06	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.08	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.88	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
120	83.85	86.92	91.58	95.70	100.62	140.23	146.57	152.21	158.95	163.65

For $n > 120$, $\chi^2_{n;\alpha} \approx n\left[1 - \frac{2}{9n} + z_\alpha\sqrt{\frac{2}{9n}}\right]^3$ where z_α is the desired percentage point for a standardized normal distribution.