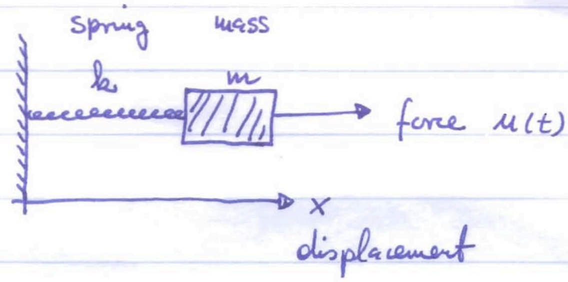
Class #12

Cross-Spectral Analysis (linear system theory)

Example:



input : force $u(t)$
output : displacement $x(t)$
system parameters : spring constant k
mass — " — m

balance of forces $\quad u(t) + u_k(t) + u_m(t) = 0$

$u_k(t) = -k\,x(t) \qquad u_m(t) = -m\,\ddot{x}(t)$ inertia = mass * acceleration
restoring force of spring

$m\ddot{x} + kx = u$ ~~Force~~ mass * acceleration (inertia) = sum of all forces (external applied + spring)

linear system $\qquad u(t) \longrightarrow \boxed{k, m} \longrightarrow x(t)$

What's the "transfer" function that transforms the input $u(t)$ to the output $x(t)$?
What's the response of the system to an input

$$X(f) = H(f) \cdot U(f)$$

for unit impulse, i.e., $u(t) = \delta(t)$, $\mathcal{F}(\delta(t)) = 1$

$\Rightarrow \quad X(f) = H(f) \cdot 1$ definition of transfer fct.

$$\mathcal{F}\left(m\ddot{x} + kx = u(t)\right)$$

$$\Rightarrow \quad \left[-m\,(2\pi f)^2 + k\right]\cdot X(f) = U(f)$$

unit impulse as input, that is $u(t) = \delta(t)$ $\Rightarrow$ $\mathcal{F}(u(t)) = 1$

$$\Rightarrow \quad \left[-m\,(2\pi f)^2 + k\right]\cdot H(f) = 1$$

$$\Rightarrow \quad H(f) = \left[k - m\,(2\pi f)^2\right]^{-1}$$

resonance for $k = m\,(2\pi f_c)^2$ or $f_c = \sqrt{\frac{k}{m}}\cdot\frac{1}{2\pi}$

Tacoma Bridge Video

instanteneous response

no phase lag (in this special case) as $H(f)$ is real, however, this is not generally the case, i.e., consider a "damping" term in the spring system: (linear friction)

$$\mathcal{F}\left(m\ddot{x} + r\dot{x} + kx = u(t)\right)$$

$$\Rightarrow \quad \left[-m\,(2\pi f^2) + rj\,(2\pi f) + k\right]\cdot H(f) = 1 \qquad u(t) = \delta(t)$$

$\Rightarrow$

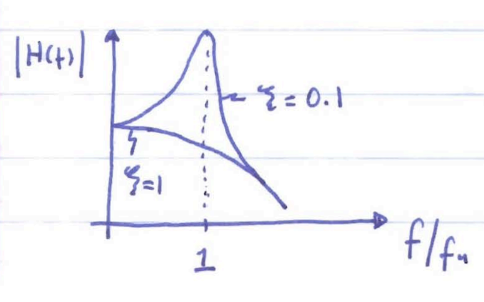
$$H(f) = \frac{1/k}{1 - (f/f_n)^2 + j\,2\xi(f/f_n)}$$

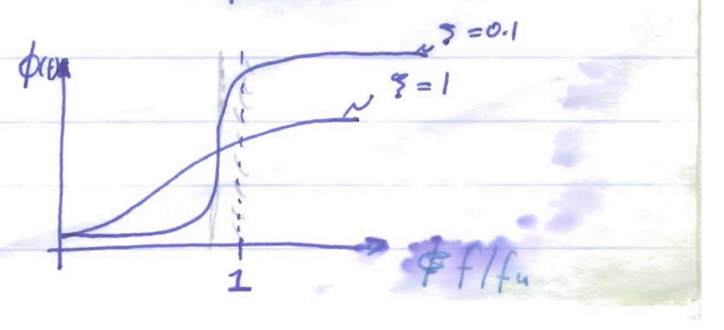
"natural" or "resonance" frequency

$$f_n = \frac{1}{2\pi}\sqrt{k/m}$$

$$\xi = r/2\sqrt{km}$$

or $\quad H(f) = |H(f)|\,e^{-j\,\phi(f)} \quad$ polar notation





Cross-spectral analysis is not really interested directly in the system but rather in it's input/output relationships, i.e., we are interested in the system's response that we infer from measurements of the

input: $x(t)$

output: $y(t)$

Cross-Covariance

[how much variance in $y(t)$ is explained by the variance of $x(t)$?]

we will construct transfer functions f[illegible] this

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)] = \iint_{-\infty}^{\infty} x(t)\, y(t+\tau)\, p(x,y)\, dx\, dy$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{\ell=1}^{N} x_\ell(t)\, y_\ell(t+\tau)$$

ergodic, stationary process $$= \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, y(t+\tau)\, d\tau$$

$R_{xy}(\tau)$ can carry both signs

$\max(R_{xy}(\tau)) \neq R_{xy}(\tau=0)$

$R_{xy}(\tau) \neq R_{xy}(-\tau)$ not an even fctn., but

$R_{xy}(\tau) = R_{yx}(-\tau)$

Cross-spectral density fctn.

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

is the ~~F~~ FFT of $R_{xy}(\tau)$ and is complex, but

$$S_{xy}(-f) = S_{xy}^{*}(f) = S_{yx}(f) \qquad \text{complex valued}$$

$$S_{xx}(-f) = S_{xx}^{*}(f) = S_{xx}(f) \qquad \text{real valued}$$

Define one sided cross-spectra

$$G_{xy} = \begin{cases} 2S_{xy} & f > 0 \\ 0 & f \le 0 \end{cases}$$

$$G_{xy}(f) = C_{xy}(f) - j\, Q(f)$$

$$= 2\int_{-\infty}^{\infty} R_{xy} \cos(2\pi f t)\, dt - 2j \int_{-\infty}^{\infty} R_{xy} \sin(2\pi f t)\, dt$$

co-spectrum   quadrature spectrum

One can also write

$$G_{xy}(f) = |G_{xy}(f)|\, e^{j\phi_{xy}(f)}$$

where

$$|G_{xy}(f)| = \sqrt{C_{xy}^2 + Q_{xy}^2}$$

$$\phi_{xy}(f) = \tan^{-1}\left(\frac{Q_{xy}(f)}{C_{xy}(f)}\right)$$

Cross-spectral analysis allows to assess the linear relation (correlation) between two time series for each frequency separately. It also allows to analyse the <u>phase</u> relations between the two series.

If $Q_{xy} = 0 \Rightarrow \phi_{xy} = 0 \Rightarrow G_{xy} = |G_{xy}| = C_{xy}$

and the two series are said to be in-phase

Def.: $\boxed{\text{Coherence } \Gamma_{xy}^2(f)}$

$$\Gamma_{xy}^2(f) = |G_{xy}(f)|^2 \Big/ G_{xx}(f) \cdot G_{yy}(f)$$

is a direct (normalised) measure of the amount of variance of $y(t)$ that is linearily related to the variance in $x(t)$. It is equivalent to the time domain cross-correlation coefficient

$$r_{12}^2 = \left(\sum_{i=1}^{N} y_1 y_2\right)^2 \Big/ \sum_{i=1}^{N} y_1^2 \sum_{i=1}^{N} y_2^2$$

In the "real" world we have estimates only, i.e.,

$$\hat{G}_{xy}(f) = \hat{C}_{xy}(f) - j\,\hat{Q}_{xy}(f)$$

$$\boxed{\hat{G}_{xy}(f) = \frac{2}{T}\,X^{*}\cdot Y}$$

$$\mathcal{F}(x(t)) = X(f) = A_x + j\,B_x = a_x\,e^{j\phi_x}$$
$$\mathcal{F}(y(t)) = Y(f) = A_y + j\,B_y = a_y\,e^{j\phi_y}$$

$$\phi_x = \tan^{-1}(B_x/A_x) \qquad a_x = (A_x^2 + B_x^2)^{1/2}$$
$$\phi_y = \tan^{-1}(B_y/A_y) \qquad a_y = (A_y^2 + B_y^2)^{1/2}$$

$$G_{xy} = \frac{2}{T}\,X^{*}Y = \frac{2}{T}\Big[\underbrace{(A_x A_y + B_x B_y)}_{\hat{C}_{xy}} - j\,\underbrace{(A_y B_x - A_x B_y)}_{\hat{Q}_{xy}}\Big]$$

Now $\hat{\phi}_{xy}(f) = \tan^{-1}\dfrac{\hat{Q}_{xy}}{\hat{C}_{xy}} = \tan^{-1}\left(\dfrac{A_y B_x - A_x B_y}{A_x A_y + B_x B_y}\right)$

$$\tan\hat{\phi}_{xy} = \frac{\frac{A_y B_x}{A_x A_y} - \frac{A_x B_y}{A_x A_y}}{1 + \frac{B_x B_y}{A_x A_y}} = \frac{\tan\hat{\phi}_x - \tan\hat{\phi}_y}{1 + \tan\hat{\phi}_x \tan\hat{\phi}_y}$$

$$= \tan(\hat{\phi}_x - \hat{\phi}_y)$$

$\hat{\phi}_{xy}$ measures the relative phase or the phase diffrence between the input $x(t)$ and the output $y(t)$

$$\hat{\phi}_{xy} = \hat{\phi}_x - \hat{\phi}_y$$

SUMMARY

A linear process between input and output is completely described by

(1) its coherence Γ_{xx}^2
(2) its phase ϕ_{xy}
(3) its amplitude or modulus $|G_{xy}|$

But we get estimates only, i.e.,

$$\hat{\Gamma}_{xy}^2 = \frac{(X^* Y) \cdot (X Y^*)}{X^* X \quad Y^* Y} \equiv 1$$

no "smoothing" done get "raw" estimates with 2 d.o.f.

We need

$$\overline{\hat{\Gamma}}_{xy}^2 = \frac{|\overline{\hat{G}_{xy}}|^2}{\overline{\hat{P}_x} \quad \overline{\hat{P}_y}} \neq 1$$

go to p. 86 for estimation of confidence levels

end class#12

Not much is known on the distributional properties of this → EMPIRICAL