$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y' = y + \varepsilon$

input — linear system — output

$$\underline{Y}(f) = \underline{H}(f) \cdot \underline{X}(f)$$ multiplication in frequency

is

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$ convolution in time

need phase information besides the gain for $H(f) = \int_{-\infty}^{+\infty} h(\tau) e^{-j2\pi f\tau}\, dt$

Consider cross-covariance — <u>unknown</u>

$$x(t)\, \underline{y(t+\tau)} = \int_{-\infty}^{+\infty} x(t)\, \underline{h(\alpha)}\, \underline{x(t+\tau-\alpha)}\, d\alpha$$

Take expected value where $x, y$ are measurement and $h$ are unknown, but fixed weights

note that integral does not involve integration of time $t$, but convolution $\alpha$

$$E[x(t)\, y(t+\tau)] = \int_{-\infty}^{+\infty} h(\alpha)\, E[x(\underline{t})\, x(\underline{t}+\underline{\tau-\alpha})]\, d\alpha$$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} h(\alpha)\, R_{xx}(\overline{\tau-\alpha})\, d\alpha$$

Take the Fourier Transform of both sides

$$\int_{-\infty}^{+\infty} R_{xy}(\tau)\, e^{-j2\pi f\tau}\, d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\alpha)\, R_{xx}(\tau-\alpha)\, d\alpha\; e^{-j2\pi f\tau}\, d\tau$$

$$S_{xy}(f) = \int\int_{-\infty}^{+\infty} h(\alpha)\, R_{xx}(q)\, \cancel{d\alpha}\, e^{-j2\pi fq}\cdot e^{-j2\pi f\alpha}\, d\alpha\, dq$$

$$q = \tau - \alpha \qquad \frac{dq}{d\tau} = 1 \qquad \tau = q + \alpha$$

$$S_{xy}(f) = \int_{-\infty}^{+\infty} h(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha \cdot \int_{-\infty}^{+\infty} R_{xx}(q)\, e^{-j2\pi fq}\, dq$$

$$S_{xy}(f) = H(f)\cdot S_{xx}(f)$$ two-side spectra

$$G_{xy}(f) = H(f)\cdot G_{xx}(f)$$ one-sided spectra

or $$|G_{xy}(f)|\, e^{-j\Theta_{xy}(f)} = |H(f)|\, e^{-j\phi(f)}\, \underbrace{G_{xx}(f)}_{\text{real}}$$

(1) $\Theta_{xy}(f) = \phi(f)$ transfer function $H$ has phase of the cross-spectrum $G_{xy}$

(2) $|H(f)| = G_{xy}(f) \big/ G_{xx}(f) = \Gamma_{xy}^2 \cdot G_{yy}$

We are talking about ideal systems, not estimates, and as long as

$$G_{xx} \neq 0 \neq G_{yy}$$

the coherence $\Gamma^2_{xy} = 1$

Otherwise, i.e., $G_{xx} = 0$ or $G_{yy} = 0$ it would be $0$

A more realistic system, however, is one that is slightly imperfect, i.e.

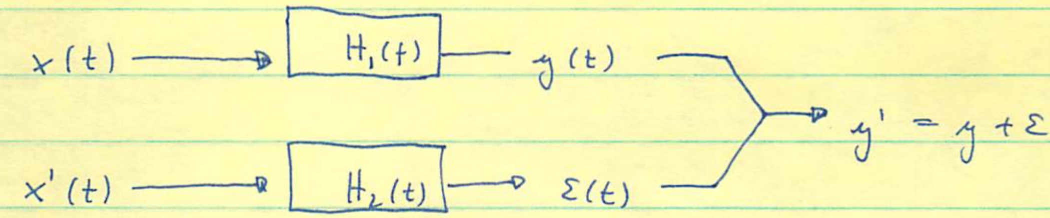
$\boxed{\text{CASE 1}}$

$$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y'(t) = y(t) + \varepsilon(t)$$

input — linear system — measured output — ideal output — output noise

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

(a) where $\varepsilon$ is noise or, possibly, contributions from another linear (or nonlinear) system:




(b) where $\varepsilon$ is random or white ~~or~~ noise (most pleasant)

(c) where $\varepsilon$ originates from nonlinear processes

If $\varepsilon(t)$ is random or pure white noise, then

$$G_{x\varepsilon}(f) = G_{y\varepsilon}(f) = 0$$

this condition can be relaxed to allow correlated output noise however the same results can be derived ~~the~~ ~~t~~ if the relevant transfer function ~~the~~ is interpreted as a least squares estimate of the transfer function that minimises the output noise

Now

$$G_{y'y'} = \frac{2}{T}(Y+\Sigma)(Y^*+\Sigma^*)$$

$$= \frac{2}{T}\left(YY^* + \underbrace{\cancel{Y\Sigma^*}}_{0} + \underbrace{\cancel{Y^*\Sigma}}_{0} + \Sigma\Sigma^*\right)$$

$$= G_{yy} + G_{\varepsilon\varepsilon}$$

$$\Gamma^2_{xy'} = \frac{|G_{xy'}|^2}{G_{xx}G_{y'y'}} = \frac{X^*(Y+\Sigma)\cdot X(Y^*+\Sigma^*)}{XX^*(Y^*+\Sigma^*)(Y+\Sigma)}$$

see p.105

$$= \frac{(G_{xy} + \cancelto{0}{G_{x\varepsilon}})\cdot(G^*_{xy} + \cancelto{0}{G^*_{x\varepsilon}})}{G_{xx}(G_{yy}+G_{\varepsilon\varepsilon})}$$

$$= \frac{|G_{xy}|^2}{G_{xx}G_{yy} + G_{xx}G_{\varepsilon\varepsilon}}$$

$$= \frac{|G_{xy}|^2 / G_{xx}G_{yy}}{1 + G_{\varepsilon\varepsilon}/G_{yy}}$$

$$= \Gamma^2_{xy} \Big/ 1 + \frac{G_{\varepsilon\varepsilon}}{G_{yy}} \xrightarrow{\Gamma^2_{xy}=1} \frac{G_{yy}}{G_{yy}+G_{\varepsilon\varepsilon}} = \frac{G_{yy}}{G_{y'y'}}$$

linear system

$$\boxed{\Gamma^2_{xy'} = \frac{|G_{xy'}|^2}{G_{xx} \cdot G_{y'y'}} = \frac{G_{yy}}{G_{y'y'}}} = \frac{\text{variance @ frequency } f \text{ that is due to the}}{\text{variance @ frequency } f \text{ that is due to both noise and linear system}}$$

where $x(t)$ measured input

$y'(t)$ measured "noisy" output, i.e., $y'(t) = y(t) + \varepsilon(t)$

$y(t)$ output due to "forcing" by input; i.e., part of the measured output that is linearily related to input

$\varepsilon(t)$ noise that is uncorrelated with both input $x(t)$ and output $y(t)$

$\Gamma^2_{xy'}$ coherence between input $x(t)$ and measured noisy output $y'(t)$

$G_{xy'}$ one-sided cross-spectra between measured input and measured noisy output

$G_{y'y'}$ one-sided auto-spectra ~~between~~ of the measured noisy input

$G_{yy}$ one-sided auto-spectra of the "clean" unmeasured output produced by the linear system; this property can not be computed directly but through the coherence fctn.

$\Rightarrow \Gamma^2_{xy'}$ is the fraction of the output variance that ~~is due to the linear system~~ can be explained by the input through ~~by~~ a linear system

examples/applications

Fig. 10 p. 486 of Munchow et al (1992) → (1) astronomical and nonlinear tides; remove the "wind" forcing

(2) barotropic vs. baroclinic tides; keep the sealevel coherent part

Fig. 3 p. 566 of Munchow (1998)

part II of class 13

"Statistics" of $\hat{\bar{\Gamma}}^2_{xy}(f)$

for large degrees of freedom d.o.f.

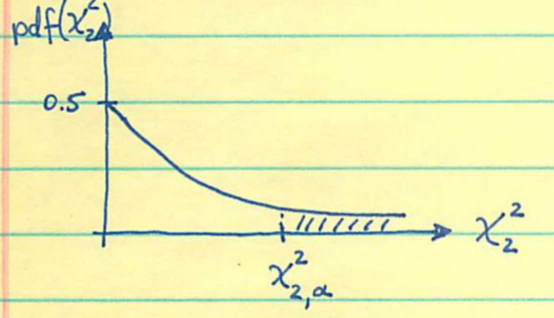
i.e., d.o.f > 40

1. Hypothesis testing; Confidence levels

Hypothesis : $x(t)$ and $y(t)$ are independent, i.e., $\hat{\bar{\Gamma}}^2_{xy} = 0$

For large d.o.f. $n$ an empirical result is that

$$n \cdot \hat{\bar{\Gamma}}^2_{xy} \approx \chi^2_2$$




there is an $\alpha$ % chance that $n\,\hat{\bar{\Gamma}}^2_{xy} > \chi^2_{2,\alpha}$

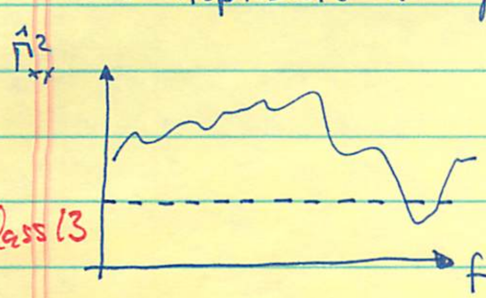
$\chi^2_{2,\,0.05} \cong 6$

$$\text{Prob}\left[\ \hat{\bar{\Gamma}}^2_{xy} > \frac{\chi^2_{2,\,\alpha=0.05}}{n}\ \right] = 0.05 = \alpha$$

→ 95% confidence level from $\frac{6}{n}$

i.e., $\hat{\bar{\Gamma}}^2 > \frac{6}{n}$ → 95% chance to be different from 0

This represents a significance level, not a confidence interval




95% level of coherence

95% chance of $\hat{\bar{\Gamma}}^2_{xy}$ to be significantly different from 0

5% — " —

end of class 13

Go to p.-92 "Multiple Input - Single Output