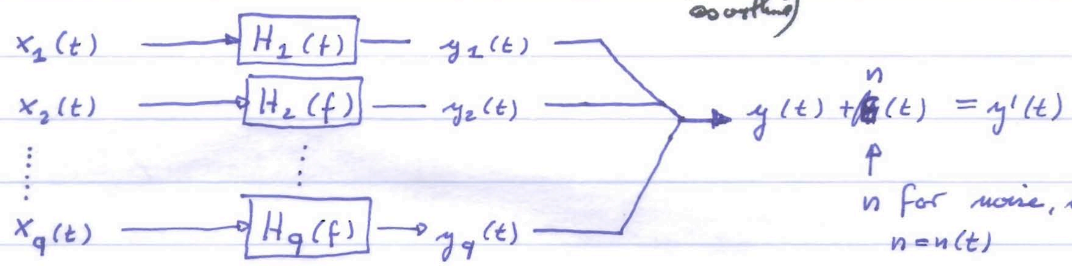
Hence we got N linear equations to solve for the N unknown signal to noise ratios $1/c_i$ $\quad i=1,2,\dots N$

input may be correlated

Multiple input / single output

(started with p. 98 as outline)

$$\left\{\begin{aligned} G_{xy} &= H \cdot G_{xx} \\ H &= G_{xy}/G_{xx} \\ \gamma^2_{xy'} &= G_{yy}/G_{y'y'} \end{aligned}\right\}$$



$$y'(t) = \sum_{k=1}^{q} y_k(t) + n(t)$$

$$\underbrace{Y_k(f)}_{\substack{\text{not measured} \\ \text{"clean" output of system } k}} = \underbrace{H_k(f)}_{\text{unknown}} \underbrace{X_k(f)}_{\text{measured input}} \qquad k=1,2,\dots q$$

$$\underbrace{Y'(f)}_{\substack{\text{measured noisy} \\ \text{output}}} = \sum_{k=1}^{q} \underbrace{H_k(f)}_{\text{unknown}} \underbrace{X_k(f)}_{\substack{\text{measured} \\ \text{input}}} + \underbrace{N(f)}_{\substack{\text{unknown} \\ \text{output noise}}}$$

$$X_i^*(f)\, Y'(f) = \sum_{k=1}^{q} H_k(f) \cdot X_k(f)\, X_i^*(f) + X_i^*(f) N(f)$$

$$\left\{ E[X_i^* Y'] = \sum_{k=1}^{q} H_k\, E[X_k X_i^*] + E[X_i N] \right\}$$ not needed.

$$\underbrace{G_{x_i y'}}_{G_{iy'}} = \sum_{k=1}^{q} H_k \underbrace{G_{x_k x_i}}_{G_{ki}} + \cancel{G_{x_i n}} = G_{x_i y}$$

$i=1,2,\dots q$

0 if $x_i(t)$ and $n(t)$ are uncorrelated

$x_i(t)$ and $n(t)$ are uncorrelated

$$\underset{\text{known}}{G_{iy'}} = G_{iy} = \sum_{k=1}^{q} \underset{\text{unknown}}{H_k}\ \underset{\text{known}}{G_{ki}} \qquad i = 1, 2, \ldots q$$

This is a set of q linear equations for the q unknowns that are $(H_1, H_2, \ldots H_q)$

$$G_{y'y'} = \frac{2}{T} Y'^{*} Y' = \frac{2}{T}\left(\sum_{k=1}^{q} H_k^* X_k^* + N\right)\cdot\left(\sum_{k=1}^{q} H_k X_k + N\right)$$

$$= \frac{2}{T}\left(\sum_{k=1}^{q}\sum_{l=1}^{q} H_k^* H_l G_{kl} + G_{nn}\right)$$

$$+ \frac{2}{T}\left(\sum_{k=1}^{q} H_k^* \underset{0}{\cancel{G_{kn}}} + \sum_{l=1}^{q} H_l \underset{0}{\cancel{G_{ln}}}\right)$$

$\boxed{\text{if output noise is uncorrelated with inputs}}$

$$\boxed{G_{y'y'} = G_{yy} + G_{nn} \qquad \text{where } G_{yy} = \sum_{k=1}^{q}\sum_{l=1}^{q} H_k^* H_l G_{kl}}$$

Now define the <u>multiple coherences</u> of the output and all the inputs as

$$\Gamma^2_{y:x} \equiv \frac{G_{yy}}{G_{y'y'}} = \frac{G_{y'y'} - G_{nn}}{G_{y'y'}} = 1 - \frac{G_{nn}}{G_{y'y'}} \leq 1$$

end of class #14

~~special case $q = 1$:~~

$$\cancel{\Gamma^2_{y:x} = \frac{G_{yy}}{G_{yy} + G_{nn}} = \frac{|H|^2 \cdot G_{11}}{G_{y'y'}} = \frac{H \cdot H^* G_{11}}{G_{y'y'}}} \quad \frac{G_{xx}}{G_{xx}}$$

~~94~~A 95A

Start with

$$\underbrace{G_{iy'}}_{\substack{\text{measured}\\ \text{calculated}}} \overset{\substack{\text{inputs } x_i \text{ uncorrelated}\\ \text{with output noise}}}{=} \underbrace{G_{iy}}_{\substack{\text{ideal}\\ \text{linear system}}} = \sum_{k=1}^{q} \underbrace{H_k}_{\text{unknown}} \underbrace{G_{ki}}_{\substack{\text{measured}\\ \text{calculated}}} \qquad i = 1, 2, \dots q$$

How to determine a property like $G_{iy'}$? Ex.: $q = 2$

$$G_{iy'} = \frac{2}{T} E[X_i^* Y'] = \frac{2}{T} E[X_i^* (H_1 X_1 + H_2 X_2 + N)]$$

$$= \frac{2}{T} E[H_1 X_i^* X_1] + \frac{2}{T} E[H_2 X_i^* X_2] + \frac{2}{T} E[X_i N]$$

$$= \frac{2}{T} H_1 E\underbrace{[X_i^* X_1]}_{} + \frac{2}{T} H_2 E\underbrace{[X_i^* X_2]}_{} + 0$$

read this "expected" value as "ensemble averaged"

$$= \bar{\hat{G}}_{i1} \cdot H_1 + \bar{\hat{G}}_{i2} H_2$$

$$= G_{i1} H_1 + G_{i2} H_2 \qquad \text{for shorthand}$$

So

$$\left.\begin{aligned} G_{1y'} &= G_{11} H_1 + G_{12} H_2 \\ \underbrace{G_{2y'}}_{\text{known}} &= \underbrace{G_{21}}_{\text{known}} \underbrace{H_1}_{\text{unknown}} + \underbrace{G_{22}}_{\text{known}} \underbrace{H_2}_{\text{unknown}} \end{aligned}\right\} \quad 2 \times 2 \text{ matrix to solve for } H_1 \text{ and } H_2$$

Go to p. ~~94~~ 97 → partial coherence

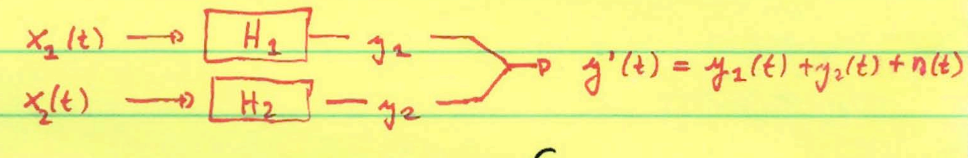
CLASS #15

example $q = 1$

$$H_1 \cdot G_{11} = G_{1y} = G_{x_1 y}$$

$$\Gamma^2_{y':x} = \frac{G_{yy}}{G_{y'y'}} = \frac{|H_1|^2 G_{11}}{G_{y'y'}} = \frac{H_1 H_1^* G_{11}}{G_{y'y'}} \frac{G_{11}}{G_{11}} = \frac{G_{1y} \cdot G_{1y}^*}{G_{y'y'} G_{11}}$$

$$= \frac{|G_{1y}|^2}{G_{y'y'} G_{11}} = \frac{|G_{1y'}|^2}{G_{y'y'} G_{11}} = \Gamma^2_{1y'} = \Gamma^2_{xy}$$



example $q = 2$

$$\Gamma^2_{y':x} = \frac{G_{yy}}{G_{y'y'}} = \frac{\left| H_1^* H_1 G_{11} + H_1^* H_2 G_{12} + H_2^* H_1 \overset{G_{21}}{\cancel{G_{22}}} + H_2^* H_2 G_{22} \right|}{G_{y'y'}}$$

or

$$G_{y'y'} \Gamma_{y':x} = \left| H_1^* \left(H_1 G_{11} + H_2 G_{12} \right) + H_2^* \left(H_1 G_{21} + H_2 G_{22} \right) \right|$$

$$= \left| H_1^* G_{1y} + H_2^* G_{2y} \right|$$

because

$$G_{1y} = H_1 G_{11} + H_2 G_{12} \quad ; \quad G_{2y} = H_1 G_{21} + H_2 G_{22}$$

which follows from

$$G_{1y'} = \frac{2}{T} E[X_1^* Y] = \frac{2}{T} E[X_1^* (H_1 X_1 + H_2 X_2 + N)]$$

$$= \frac{2}{T} E[H_1 X_1^* X_1] + \frac{2}{T} E[H_2 X_1^* X_2] + \frac{2}{T} E[X_1^* N]$$

$$G_{1y'} = \frac{2}{T} H_1 E[X_1^* X_1] + \frac{2}{T} H_2 E[X_1^* X_2] + \frac{2}{T} E[X_1^* N]$$

$$G_{1y'} = H_1 G_{11} + H_2 G_{12} + G_{1n} \xrightarrow{G_{1n}=0} = G_{1y}$$

$$G_{2y'} = H_1 G_{21} + H_2 G_{22} + G_{2n} \xrightarrow{G_{2n}=0} = G_{2y}$$

If and only if the output noise is uncorrelated or independent of the input functions, i.e., $G_{1n} = G_{2n} = 0$, can we solve for the transfer functions H_1 and H_2 which after some algebra are

if uncorrelated inputs ← special case
$G_{12}=0$

keep for p.99
Homework #3 in 2009

$$H_1 = \frac{G_{1y}\left(1 - \frac{G_{12} G_{2y}}{G_{22} G_{1y}}\right)}{G_{11}\left(1 - \Gamma_{12}^2\right)} \;\xrightarrow{G_{12}=0}\; = \frac{G_{1y}}{G_{11}} \qquad \Gamma_{12}^2 = \Gamma_{x_1 x_2}^2 = \frac{|G_{12}|^2}{G_{11}^2 G_{22}^2}$$

if inputs are not correlated
$G_{12}=0$

$$H_2 = \frac{G_{2y}\left(1 - \frac{G_{21} G_{1y}}{G_{11} G_{2y}}\right)}{G_{22}\left(1 - \Gamma_{12}^2\right)} \;\xrightarrow{G_{12}=0}\; = \frac{G_{2y}}{G_{22}}$$

which gives

if inputs are not correlated
$G_{12}=0$ & $G_{iy} = H_i G_{ii}$

$$\Gamma_{y:x}^2 = \frac{|H_1^* G_{1y}|}{G_{y'y'}} + \frac{|H_2^* G_{2y}|}{G_{y'y'}} = \frac{G_{1y}^* G_{1y}}{G_{11} G_{y'y'}} + \frac{G_{2y}^* G_{2y}}{G_{22} G_{y'y'}} =$$

$$= \frac{G_{yy}}{G_{y'y'}}$$

What to do if input are related?

$$= \frac{|G_{1y}|^2}{G_{11} G_{y'y'}} + \frac{|G_{2y}|^2}{G_{22} G_{y'y'}} = \Gamma_{1y'}^2 + \Gamma_{2y'}^2$$

this is true only for the special case of uncorrelated inputs