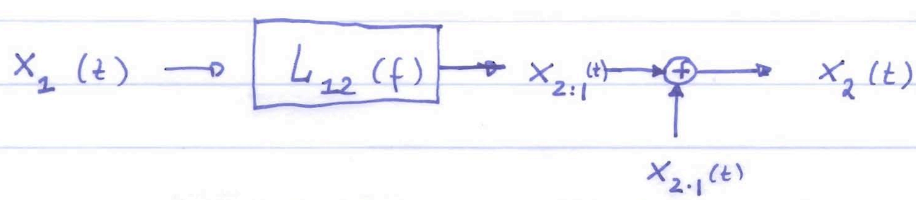
PARTIAL or CONDITIONAL COHERENCE / SPECTRA

notation $x_{2\cdot 1}(t)$ represents the part of the record $x_2(t)$ that is NOT correlated with $x_1(t)$.

decompose $x_2(t)$ into two parts, i.e.,

$$x_2(t) = x_{2:1}(t) + x_{2\cdot 1}(t)$$

measured input	computed part of input x_2 that is ~~coherent~~ correlated with x_1	computed part of input x_2 that is NOT correlated with x_2

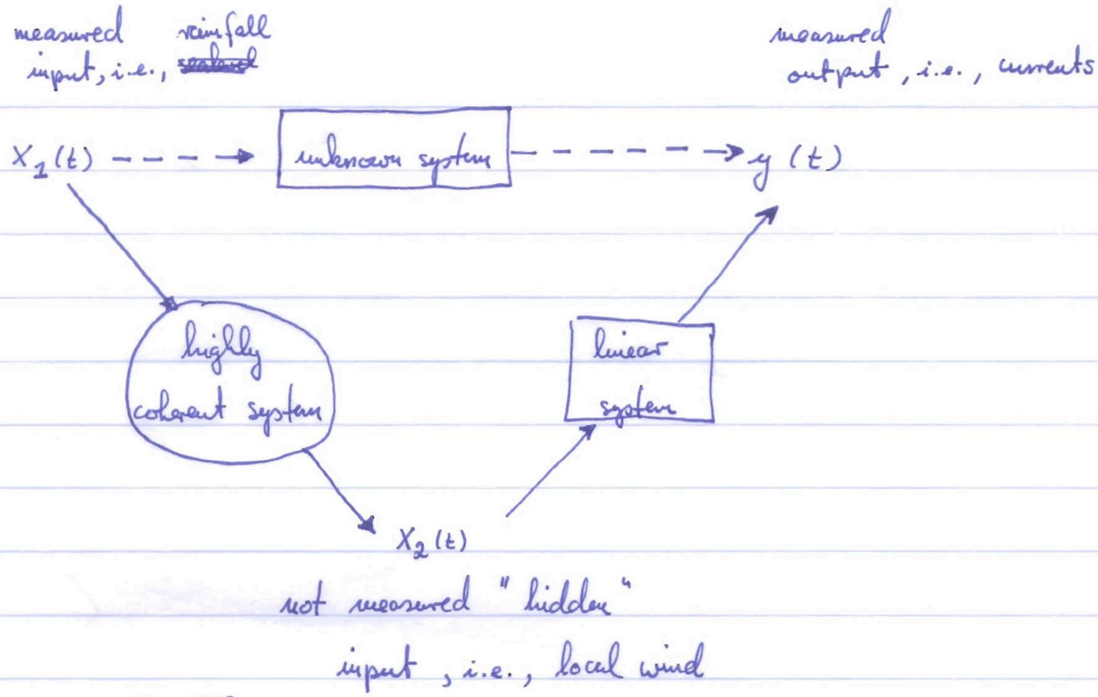


frequency domain

$$X_{2\cdot 1}(f) = X_2(f) - L_{12} X_1(f)$$

$$= X_2 - \frac{G_{12}}{G_{11}} X_1$$

"The constant parameter system $L_{12}(f)$ represents the optimum linear system to predict $x_2(t)$ from $x_1(t)$ ..." Bendat + Piersol (1986, p.216)

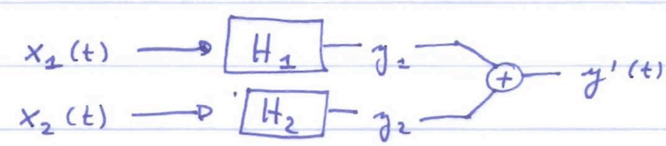


erroneous

artificially high coherence between rainfall and ocean current <u>because</u> the rainfall correlates strongly with the wind field which, incidentally, also correlates well with ocean currents

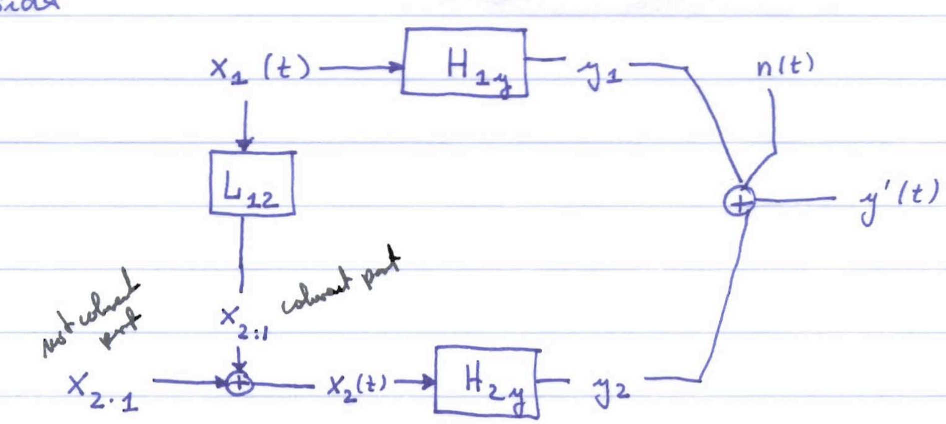
→ pitfalls of spectral or cause-effect analyses

instead of

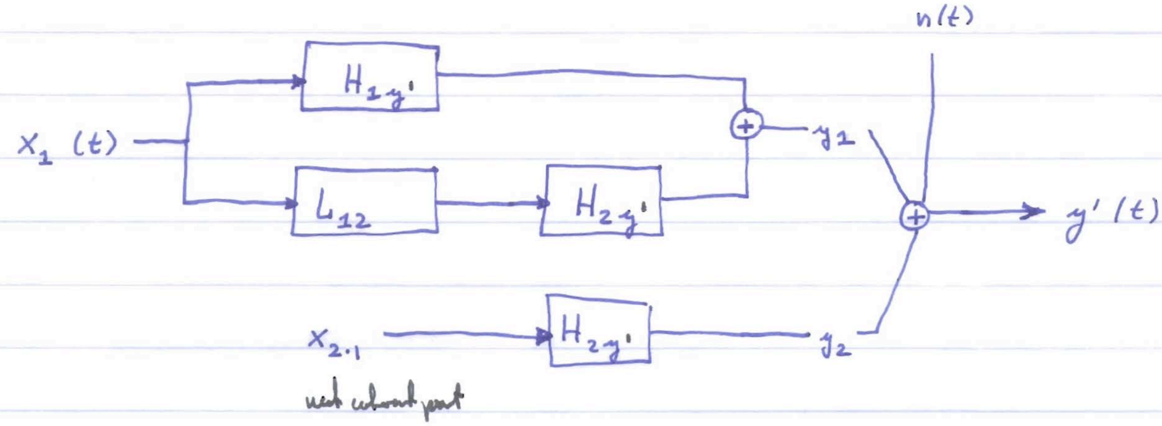


$$X_2 = \underbrace{X_{2\cdot 1}}_{\text{incoherent with } x_1} + \overbrace{X_{2:1}}^{\text{coherent with } x_1}$$

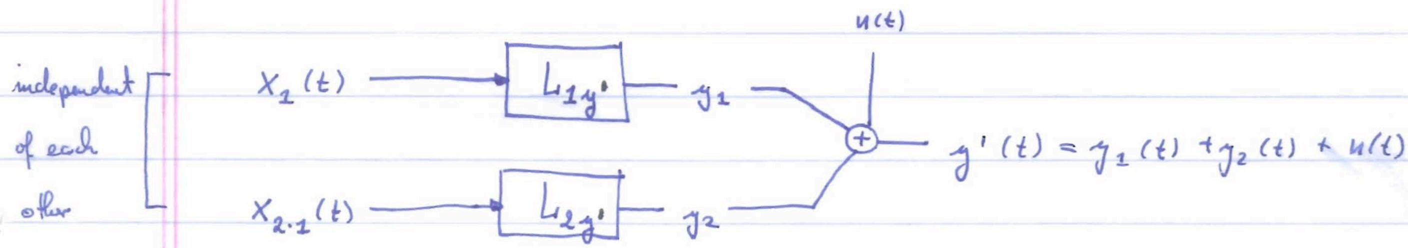
consider



which is equivalent to



which is equivalent to



where

$$L_{1y'} = H_{1y'} + L_{12} \cdot H_{2y'} = G_{1y'} / G_{11}$$

$$L_{2y'} = H_{2y'} = G_{2y' \cdot 1} / G_{22 \cdot 1}$$

from p. 95

↑ and #15

represent a system where the input functions $x_1(t)$ and $x_{2\cdot 1}(t)$ are independent, hence

$$Y'(f) = L_{1y'}(f) \cdot X_1(f) + L_{2y'}(f)\, X_2(f) + N(f)$$

and

$$L_{1y'} = G_{1y'} / G_{11} \quad \text{and} \quad L_{2y'} = G_{2y' \cdot 1} / G_{22 \cdot 1}$$

where $G_{2y'\cdot 1}(f) \hat{=}$ cross-spectra between $x_{2\cdot 1}(t)$ and $y'(t)$

$G_{22\cdot 1}(f) \hat{=}$ auto-spectra of $x_{2\cdot 1}(t)$

Conditional or partial auto-spectra

remember that $\quad x_2(t) = x_{2:1}(t) + x_{2\cdot 1}(t)$

$x_{2:1}$: part of x_2 correlated with x_2

$x_{2\cdot 1}$: part of x_2 NOT correlated with x_1

not correlated

$$X_2(f) = X_{2:1}(f) + X_{2\cdot 1}(f)$$

$$X_{2:1} = L_{12}\, X_1 \quad \rightarrow \quad X_{2\cdot 1} = X_2 - L_{12}\, X_1$$

L_{12}: transfer function of input-1 to input-2

$$= X_2 - \frac{G_{12}}{G_{11}}\, X_1$$

$x_1 \rightarrow \boxed{L_{12}} \rightarrow x_2$

$L_{12} = \dfrac{G_{12}}{G_{11}}$

$$G_{22} = G_{22:1} + G_{22\cdot 1}$$

$G_{22:1}$: coherent spectrum; $G_{22\cdot 1}$: incoherent spectrum

coherent spectrum $\quad G_{22:1} = |L_{12}|^2\, G_{11} = \dfrac{G_{12}\cdot G_{12}^*}{G_{11}\cdot G_{11}} \cdot G_{11} \cdot \dfrac{G_{22}}{G_{22}} = \dfrac{|G_{12}|^2}{G_{11}\, G_{22}} \cdot G_{22}$

$$\boxed{G_{22:1} = \Gamma_{12}^2 \cdot G_{22}}$$

incoherent spectrum $\quad G_{22\cdot 1} = G_{22} - G_{22:1} = G_{22} - \Gamma_{12}^2\, G_{22}$

$$\boxed{G_{22\cdot 1} = \left(1 - \Gamma_{12}^2\right) G_{22}}$$

or explicitly $\underbrace{\frac{2}{T} E\left[X_{2\cdot1}^{*} \cdot Y \right]}$

$$G_{2y\cdot1} = \frac{2}{T} E\Big[\underbrace{(X_2^* - L_{12}^* X_1^*)}_{} \cdot Y \Big]$$

FT of part of the record $x_2(t)$
that is NOT coherent with
$x_1(t)$

why can this be pulled out of the $E[\cdot]$ operator?

$$= \frac{2}{T} E\left[X_2^* Y \right] - L_{12}^* \frac{2}{T} E\left[X_1^* Y \right]$$

$$= G_{2y} - \frac{G_{12}^*}{G_{11}} \cdot G_{1y}$$

homework #3

$$\boxed{G_{2y\cdot1} = G_{2y} - \frac{G_{21}}{G_{11}} \cdot G_{1y}} = G_{2y'\cdot1} \quad \text{for noise uncorrelated with inputs}$$

$$= G_{2y'} - \frac{G_{21}}{G_{11}} \cdot G_{1y'}$$

for the special case that $y(t) = x_2(t)$

$$G_{22\cdot1} = G_{22} - \frac{G_{21}}{G_{11}} \cdot G_{12} = G_{22} - \frac{G_{12}^* G_{12}}{G_{11}} \frac{G_{22}}{G_{22}}$$

$$= G_{22} - \frac{|G_{12}|^2}{G_{11} G_{22}} \cdot G_{22} = G_{22}\left(1 - \Gamma_{12}^2\right)$$

or for $x_2(t) = y'(t)$

$$G_{y'y'\cdot1} = G_{y'y'} - \frac{G_{y'1} \cdot G_{1y'}}{G_{11}} = G_{y'y'} - \frac{G_{1y'}^* G_{1y'}}{G_{11}} \cdot \frac{G_{y'y'}}{G_{y'y'}} = G_{y'y'}\left(1 - \Gamma_{1y'}^2\right)$$

Partial (conditioned) coherence functions

$$G_{y'y'} = G_{y_1y_1} + G_{y_2y_2} + G_{nn}$$

$$= |L_{1y'}|^2 G_{11} + |L_{2y'}|^2 G_{22\cdot 1} + G_{y'y'\cdot 1,2}$$

this is the part of the measured output y' that is NOT coherent with either x_1 or x_2

call it "noise"

$$= \frac{G_{1y'} G^*_{1y'}}{\cancel{G_{11}} G^*_{11}} \cancel{G_{11}} + \frac{G_{2y'\cdot 1} G^*_{2y'\cdot 1}}{\cancel{G_{22\cdot 1}} G^*_{22\cdot 1}} \cancel{G_{22\cdot 1}} + G_{nn}$$

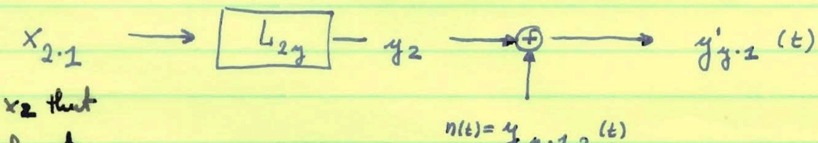
$$= \frac{G_{1y'} G^*_{1y'}}{G^*_{11}} \cdot \frac{G_{y'y'}}{G_{y'y'}} + \frac{G_{2y'\cdot 1} G^*_{2y'\cdot 1}}{G^*_{22\cdot 1}} \frac{G_{y'y'\cdot 1}}{G_{y'y'\cdot 1}} + G_{nn}$$

$$= \Gamma^2_{1y'} \cdot G_{y'y'} + \Gamma^2_{2y'\cdot 1} G_{y'y'\cdot 1} + G_{nn}$$

ordinary ~~coherence~~ spectra — conditioned ~~coherence~~ spectra — noise spectra

skip

(7.10)

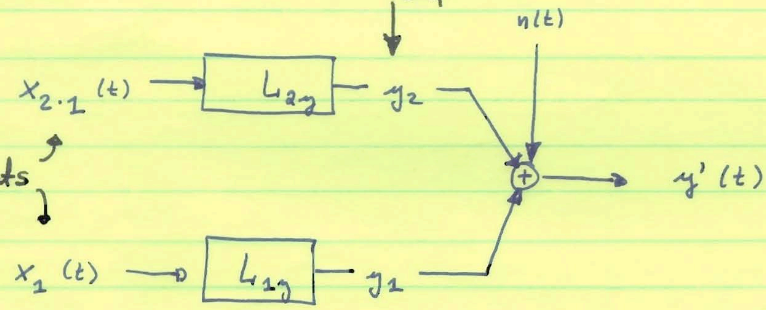


contains output noise

part of x_2 that is incoherent with x_1

does NOT contain output noise

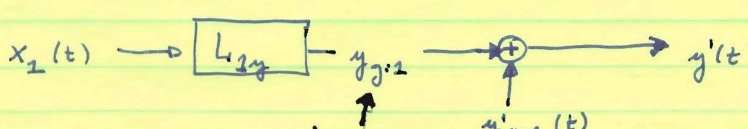
(7.8)



uncorrelated inputs

contains output noise

(7.9)



does NOT contain output noise

output that is NOT correlated with input 1

The "conditioned" system has exactly the same form as the "ordinary" system as both the input and output in the "ordinary" system are replaced by their "conditioned" equivalents. The "conditioned" input does not go to the raw output but to the "conditioned" output. Nevertheless, the cross-spectra

$$\cancel{\frac{T}{2} G_{2y\cdot 1}} = E[\,X^*_{2\cdot1}\, Y_{y\cdot1}\,] = E[\,X^*_{2\cdot1}\cdot Y\,] = \cancel{\frac{T}{2} G}$$

i.e., the "conditioned" input in the frequency domain automatically guarantes the "conditioned" output because

$$E[\,X^*_{2\cdot1}\, Y_{y\cdot1}\,] = E[\,X^*_{2\cdot1}\,(Y - \frac{G_{1y}}{G_{11}} X_1)\,]$$

$$= E[\,X^*_{2\cdot1}\, Y\,] - \frac{G_{1y}}{G_{11}}\, E[\,X^*_{2\cdot1}\cdot X_1\,]$$

$$= E[\,X^*_{2\cdot1}\, Y\,] - \frac{G_{1y}}{G_{11}}\cdot 0$$

The noise spectra, however

$$G_{nn} \equiv G_{y'y'\cdot1,2} = G_{y'y'\cdot1} - G_{y_2 y_2}$$

$$\Gamma^2_{2y'\cdot1}\cdot \overbrace{G_{y'y'\cdot1}}^{G_{y'y'}(1-\Gamma^2_{1y'})}\ \text{(p. 102)}$$

$$= G_{y'y'\cdot1} - \overbrace{|L_{2y'}|^2\, G_{22\cdot1}}$$

$$= G_{y'y'\cdot1}\,(1 - \Gamma^2_{2y\cdot1}) = \underbrace{G_{y'y'}(1-\Gamma^2_{1y'})}_{G_{y'y'\cdot1}}\,(1-\Gamma^2_{2y':1})$$

and thus

$$\Gamma^2_{y:x} = \frac{G_{yy}}{G_{y'y'}} = 1 - \frac{G_{uu}}{G_{y'y'}} = 1 - \left(1 - \Gamma^2_{1y'}\right)\left(1 - \Gamma^2_{2y'\cdot 1}\right)$$

→ Gave free example discussing Münchow et al (1992) approach to detect tidal rectificated flows of NJ/DE

Summary spectral analysis

$$G_{xx} = 2\cdot S_{xx} = 2\int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi f\tau}\, d\tau \approx \frac{2}{T}\left|\int_0^T x(t)\, e^{-j2\pi f t}\, dt\right|^2 = \frac{2}{T} X^*\cdot X$$

or

$$\hat{G}_{xx} = \frac{2}{T} E[X^* X]$$ auto-spectra

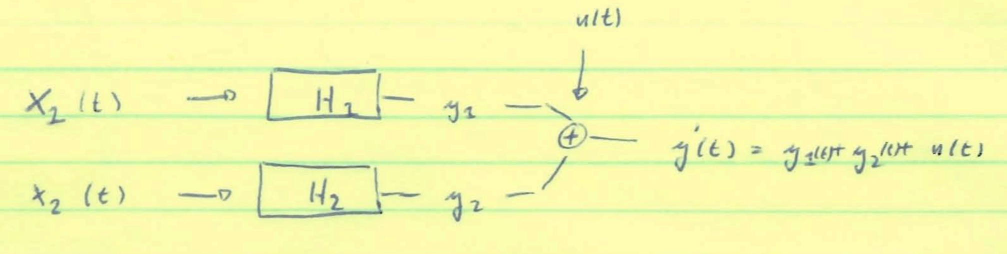
$$\hat{G}_{xy} = \frac{2}{T} E[X^* Y]$$ cross-spectra

$$\hat{\Gamma}^2_{xy} = \frac{|\hat{G}_{xy}|^2}{\hat{G}_{xx}\hat{G}_{yy}} = \frac{G_{yy}}{G_{y'y'}} = \frac{E[X^*Y\, XY^*]}{E[X^*X]\cdot E[X^*Y]}$$ coherence, i.e., variance explained by a linear input/output system

$$\hat{H} = \hat{G}_{xy} / \hat{G}_{xx}$$, i.e., $Y(f) = H(f)\cdot X(f)$ transfer function 1 input / 1 output

$$\hat{H}_k = \hat{G}_{xy} \cdot G_k$$

$$H_k = G^{-1}_{ki} \cdot G_{iy}$$, i.e., $Y = H_k \cdot X_k$ transfer functions multiple input / 1 output



$$Y(f) = Y_1(f) + Y_2(f) + N(f)$$

$$Y(f) = H_1 X_1(f) + H_2 X_2(f) + N(f)$$

$\Rightarrow$

$$N(f) = Y - H_1 X_1 - H_2 X_2$$

before we $\underline{\text{always}}$ assumed that $G_{1n} = G_{2n} = 0$, however, here we do NOT assume that, but we want to minimize the noise spectra G_{nn}

$\Rightarrow$

$$G_{nn}(f) = \frac{2}{T} E[N^* N] = G_{yy} - H_1 G_{y1} - H_2 G_{y2}$$
$$- H_1^* G_{1y} + H_1^* H_1 G_{11} + H_1^* H_2 G_{12}$$
$$- H_2^* G_{2y} + H_1 H_2^* G_{21} + H_2 H_2^* G_{22}$$

We do not know H_1, H_2 yet, however, we want to find those H_1 and H_2 that minimize the noise spectra $G_{nn} = G_{nn}(f) = G_{nn}(H_1, H_2, f)$

$$\frac{\partial G_{nn}}{\partial H_1^*} = 0 \quad \Rightarrow \quad -G_{1y} + H_1 G_{11} + H_2 G_{12} = 0$$

$$\Rightarrow \quad \boxed{G_{1y} = H_1 G_{11} + H_2 G_{12}}$$

same equation as if $G_{1n} = 0$

$$\frac{\partial G_{nn}}{\partial H_2^*} = 0 \quad \Rightarrow \quad -G_{2y} + H_1 G_{21} + H_2 G_{22} = 0$$

$$\Rightarrow \quad \boxed{G_{2y} = H_2 G_{22} + H_1 G_{21}}$$

same equation as if $G_{2n} = 0$

$$\frac{\partial G_{nn}}{\partial H_2} = 0 \quad \Rightarrow \quad -G_{y2} + H_1^* G_{12} + H_2^* G_{22}$$

$$\Rightarrow \quad G_{y2} = H_1^* G_{12} + H_2^* G_{22} \quad \Rightarrow G_{2y} = H_2 G_{22} + H_1 G_{21}$$