Start #17 (Nov. 12)

- RANDOM
  - Stationary
    - spectral techniques
  - Non-stationary
- DETERMINISTIC
  - Periodic
    - least square function fitting
      - time
      - time + space
  - Non-periodic

Tidal sealevel and velocity oscillations are prime examples of deterministic signals that generally can be predicted well into the future as the forcing takes place at <u>discrete</u> and <u>known</u> astronomical frequencies.

Often geophysical phenomena (sun-spot cycles, glacial cycles, geological cycles) are semi-deterministic, the frequencies of oscillations are not exactly known, but cycles are present nevertheless

↳

- use spectral methods to find the frequency of a signal, then
- assume that the <u>frequency</u> is known and discrete, and
- apply least square function fitting using the now "known frequencies"

Method of least squares

Model : $y(t) = a + bt$ 2 free parameters (a, b)

Measurements: $y_i = y(t_i) = a + b t_i$

misfit between model and measurement $y_i(t_i)$, $i = 1, 2, 3, \ldots N$

$$\varepsilon^2 = \sum_{i=1}^{N} (y_i - y)^2 = \sum_{i=1}^{N} \left[y_i - (a + b t_i) \right]^2 = \varepsilon^2(a, b)$$

minimize the mean square error (the misfit), i.e.,

$$\frac{\partial \varepsilon^2}{\partial (a, b)} = 0 \quad , \text{ i.e..}$$

$$\frac{\partial \varepsilon^2}{\partial a} = 0 \quad \Rightarrow \frac{\partial}{\partial a} \sum_{i=1}^{N} \left[y_i - (a + b t_i) \right]^2 = 0$$

$$\Rightarrow \quad 2 \sum_{i=1}^{N} [\, y_i - a + b t_i \,] \cdot (-1) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} (a + b t_i) = 0$$

$$\frac{\partial \varepsilon^2}{\partial b} = 0 \quad \Rightarrow \quad 2 \sum_{i=1}^{N} \left(y_i - (a + b t_i) \right) \cdot (-t_i) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{N} y_i \cdot t_i - \sum_{i=1}^{N} (a + b t_i)\, t_i = 0$$

$$\sum_{i=1}^{N} y_i - \sum_{i=1}^{N} a - \sum_{i=1}^{N} b t_i = 0$$

$$\sum_{i=1}^{N} y_i - N \cdot a - b \sum_{i=1}^{N} t_i = 0$$

or with $[\cdot] \equiv \sum_{i=1}^{N} \cdot$

$$\left.\begin{array}{l} [y] - Na - b[t] = 0 \\ [yt] - a[t] - b[t^2] = 0 \end{array}\right\} \begin{pmatrix} N & [t] \\ [t] & [t^2] \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} [y] \\ [yt] \end{pmatrix}$$

A least-square problem can always be written in the form $\underset{\sim}{A} \cdot \vec{x} = \vec{d}$

where $\underset{\sim}{A}$ is an $(m \times m)$ matrix where m is the number of free parameters

$\vec{x}$ is a vector of length m containing the "unknown" free parameters

$\vec{d}$ is a data vector

$$\vec{x} = \underset{\sim}{A}^{-1} \cdot \vec{d}$$

gives the solution of the least-square problem

Harmonic (tidal) analysis

~~y(t~~) model $\quad y(t) = A_0 + \sum_{m=1}^{M} b_m \cos(\omega_m t) + c_m \sin(\omega_m t)$

here ω_m are known tidal frequencies,
y is either a sea level height or a velocity component,
t is time, and
(A_0, b_m, c_m) are harmonic coefficients to be found

Let's assume that the data are equally spaced (such as those from an NOS tide gauge or a current meter mooring), i.e., we have measurements

$$y_\nu = y_{-n}, y_{-n+1}, y_{-n+2}, \cdots, y_0, y_1, y_2, \cdots y_{n-1}, y_n$$

to be fitted to

$$y(t = \nu \cdot \Delta t) = A_0 + \sum_{r=1}^{M} A_r \cos(\omega_r \nu \Delta t) + B_r \sin(\omega_r \nu \cdot \Delta t)$$

The mean square error ε^2 is

$$\varepsilon^2 = \sum_{\nu=-n}^{n} \left(y_\nu - y(\nu \cdot \Delta t) \right)^2 = \text{minimum}$$

$$= \sum_{\nu=-n}^{n} \left\{ y_\nu - \left[A_0 + \sum_{r=1}^{M} A_r \cos(\omega_r \nu \Delta t) + B_r \sin(\omega_r \nu \Delta t) \right] \right\}^2$$

Hence

$$\frac{\partial \varepsilon^2}{\partial A_0} = \frac{\partial \varepsilon^2}{\partial A_\tau} = \frac{\partial \varepsilon^2}{\partial B_\tau} = 0 \qquad \tau = 1, 2, \ldots M$$

give $(2M+1)$ equations for the $(2M+1)$ unknowns that are $\vec{x} = (A_0, A_\tau, B_\tau, \tau = 1, 2, \ldots M)$. These equations can be written as

$$\underset{\sim}{F} \cdot \vec{x} = \vec{D}$$

and ~~result in the~~

$$\vec{x} = \underset{\sim}{F}^{-1} \cdot \vec{D}$$

where

$$\vec{D} = \begin{pmatrix} \sum_{\nu=-n}^{n} y_\nu \\ \sum_{\nu=-n}^{n} y_\nu \cos(\omega_1 \nu \cdot \Delta t) \\ \vdots \\ \sum_{\nu=-n}^{n} y_\nu \cos(\omega_M \nu \cdot \Delta t) \\ \sum_{\nu=-n}^{n} y_\nu \sin(\omega_1 \nu \cdot \Delta t) \\ \vdots \\ \sum_{\nu=-n}^{n} y_\nu \sin(\omega_M \nu \cdot \Delta t) \end{pmatrix}$$

where

$$\underset{\sim}{F} = \begin{pmatrix} N & h_1 & h_2 & \ldots & h_M & 0 & \ldots & 0 \\ h_1 & f_{11} & f_{21} & \ldots & f_{M1} & 0 & \ldots & 0 \\ h_2 & f_{12} & f_{22} & \ldots & f_{M2} & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & & \vdots & & \\ h_M & f_{1M} & & & f_{MM} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & g_{11} & \ldots & g_{M1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & g_{1M} & \ldots & g_{MM} \end{pmatrix}$$

This is a symmetric $(2M+1 \times 2M+1)$ matrix ~~where~~ which is <u>independent</u> of the measurements y_ν

$$h_i = S(\omega_i)$$

$$F_{ij} = S(\omega_i - \omega_j) + S(\omega_i + \omega_j) = F_{ji}$$

$$G_{ij} = S(\omega_i - \omega_j) - S(\omega_i + \omega_j) = G_{ji}$$

and

$$S(\omega) = \frac{\sin\{[(2n+1)\cdot\omega/2]\}}{\sin(\omega/2)} = S(-\omega)$$

$$f_{ij} = F_{ij} * 0.5 \qquad S(\omega=0) = N$$

$$g_{ij} = G_{ij} * 0.5$$

$N = 2n+1$ is the # of data

There is <u>exten</u>sive algebra in this matrix which has both pleasing symmetries and appealing simplicity; use has been made of the following relations that depend crucially on the use of "central" time:

$$[\cdot] \equiv \sum_{\nu=-n}^{n} \cdot$$

$$[\cos(\omega_r \nu \Delta t)] = S(\omega_r) \qquad 2\,\cancel{[\cos(\omega_r \nu\Delta t)\sin(\omega_s \nu\Delta}t)$$

$$[\sin(\omega_r \nu \Delta t)\cos(\omega_s \nu \Delta t)] = 0$$

$$[\sin(\omega_r \nu \Delta t)] = 0$$

$$2[\cos^2(\omega_r \nu \Delta t)] = N + S(2\omega_r)$$

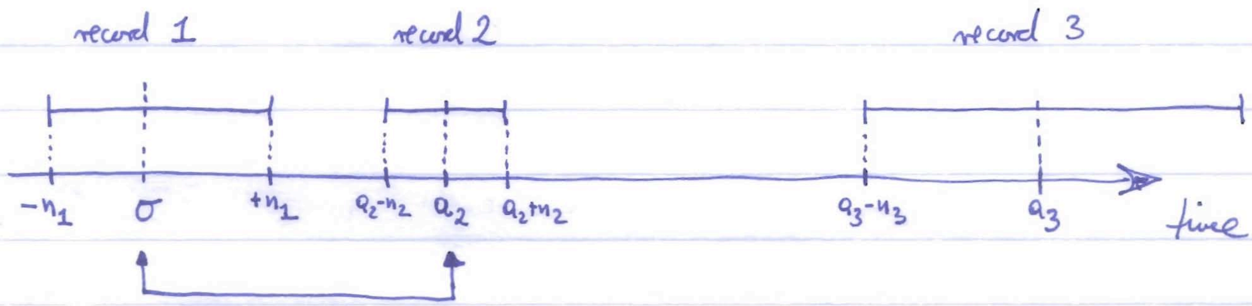
$$2[\sin^2(\omega_r \nu \Delta t)] = N - S(2\omega_r)$$

and

$$2\left[\cos(\omega_r \nu \Delta t)\cos(\omega_s \nu \Delta t)\right] = S(\omega_r - \omega_s) + S(\omega_r + \omega_s) = F_{rs}$$

$$2\left[\sin(\omega_r \nu \Delta t)\sin(\omega_s \nu \Delta t)\right] = S(\omega_r - \omega_s) - S(\omega_r + \omega_s) = G_{rs}$$

Let's now consider data that is not equally spaced, i.e., data from the same mooring location, but different deployments from turn around, gaps due to instrument failure, gaps due to biological fouling, etc.



distance between the CENTRAL points of the 2 series

data in each segment is equally spaced, i.e.,

$$y_\nu = y_{-n_1},\ y_{-n_1+1},\ \dots,\ y_0,\ y_1,\ \dots,\ y_{n_1-1},\ y_{n_1}$$

$a \equiv a_2$ $$y_{a-n_2},\ y_{a-n_2+1},\ \dots,\ y_a,\ y_{a+1},\ \dots,\ y_{a+n_2-1},\ y_{a+n_2}$$

$\vdots$

shall be fitted to .

$$y(t = \nu \cdot \Delta t) = A_0 + \sum_{r=1}^{M} A_r \cos(\omega_r \nu \Delta t) + B_r \sin(\omega_r \nu \Delta t)$$

The least square error now becomes

$$\varepsilon^2 = \sum_{\nu=-n_1}^{n_1} \left(y_\nu - y(\nu \cdot \Delta t) \right)^2 + \sum_{\nu=-n_2}^{n_2} \left(y_{a+\nu} - y((a+\nu)\Delta t) \right)^2$$

Same procedure as before, i.e.

$$\frac{\partial \varepsilon^2}{\partial A_0} = \frac{\partial \varepsilon^2}{\partial A_r} = \frac{\partial \varepsilon^2}{\partial B_r} = 0 \quad \rightarrow \quad \underset{\sim}{F} \cdot \vec{x} = \vec{D}$$

$$D = \begin{pmatrix} \sum_{\nu=-n_1}^{n} \cdot + \sum_{\nu=-n_2}^{n_2} \cdot \\ \vdots \end{pmatrix}$$

and 10 days of algebra as a second year PhD graduate student

gives

$$\underset{\sim}{F} = \begin{pmatrix} N^1 + N^2 & h_1^1 + h_1^2 & h_2^1 + h_2^2 & \dots & & \\ h_1^1 + h_1^2 & f_{11}^1 + f_{11}^2 & & & & \\ h_2^1 + h_2^1 & & f_{22}^1 + f_{22}^2 & & & \\ \vdots & & & \ddots & & \\ & & & & g_{11}^1 + g_{11}^2 & \\ & & & & & g_{22}^1 + g_{22}^2 \\ & & & & & & \ddots \end{pmatrix}$$

For a single, centered time series the matrix elements in $\boxed{}$ were identical zero. However, now they become filled with sums of functions which ALL have a multiplicative factor of $\sin(a\omega_\sigma)$ $\sigma = 1, 2, \ldots M$

hence

$$\lim_{a \to 0} \boxed{} = 0$$

reference : Munchow, Masse, and Garvine (1992). Astronomical and nonlinear tidal currents in a coupled estuary shelf system. Cont. Shelf. Res., 12, 471-498.

also error analysis and discussion of signal detection
→ programme GAPPY2.F available by this author

next: spatial data from shipborne ADCP surveys which are necessarily biased by tidal currents

reference : Candela, Beardsley, and Limeburner (1992). Separation of tidal and subtidal currents in ship-mounted acoustic Doppler current profiler observations, JGR, 97, 769-788.