class #18 (Nov. 16)

$$u(t) = A_0 + \sum_{r=1}^{M} B_r \cos(\omega_r t) + C_r \sin(\omega_r t)$$

$$u(x,y,z,t) = A_0(x,y,z) + \sum_{r=1}^{M} B_r(x,y,z) \cos(\omega_r t) + C_r(x,y,z) \cos(\omega_r t)$$

where

$$A_0(x,y,z) = \sum_{k=1}^{K} \alpha_k f_k(x,y,z)$$

$$B_r(x,y,z) = \sum_{l=1}^{L} \beta_{kr} g_l(x,y,z)$$

β_{lr}

$$C_r(x,y,z) = \sum_{l=1}^{L} \gamma_{kr} g_l(x,y,z)$$

γ_{lr}

and $\vec{x} = (\alpha_k, \beta_{lr}, \gamma_{lr})$ are $K + 2 \cdot M \cdot L$ free constant parameters to be found by minimizing the least square error ε^2 and

$f_k(x,y,z)$ and $g_{lr}(x,y,z)$ are explicitely specified "base" functions to be choosen subjectively. Anything goes here

As the ~~eq~~ model equation is linear in the parameters $\vec{x}$, we still have a problem of the form

$$\underset{\sim}{F} \cdot \vec{x} = \vec{D}$$

The trick (and hard work) is to "set-up" a proper matrix $\underset{\sim}{F}$ that has $(K+2ML) \times (K+2ML)$ elements.

In ADCP surveys people used

(a) polynomials (Candela et al., 1992 ; JGR)

$$A_0(x,y) = \sum_{j=0}^{J_0} \sum_{k=0}^{j} \alpha_l \, x^{j-k} y^k$$

J_0, J_1 is the degree of the polynomial

$l = (j-k, k)$ indices for the free parameter

$$B_r(x,y) = \sum_{j=0}^{J_1} \sum_{k=0}^{j} \beta_{lr} x^{j-k} y^k$$

$$C_r(x,y) = \sum_{j=0}^{J_1} \sum_{k=0}^{j} \gamma_{lr} x^{j-k} y^k$$

(b) Ekman layer solutions and linear polynomial (Münchow et al., 1992 CSR)

$$B_r(x,z) = (\beta_1 + \beta_2 x)\cos(\xi)\cosh(\xi) + (\beta_3 + \beta_4 x)\sin(\xi)\sinh(\xi)$$
$$C_r(x,z) = (\gamma_1 + \gamma_2 x)\cos(\xi)\cosh(\xi) + (\gamma_3 + \gamma_4 x)\sin(\xi)\sinh(\xi)$$

where $\xi = z/\delta_E = z/\sqrt{2A_v/f}$

A_v vertical viscosity

f coriolis parameter

(c) biharmonic (spatial) splines (Sandwell et al., 1992; JGR)
(Wong and Muchow, 1995; CSR)

$$A_0(x,y) = \sum_{k=1}^{K} \alpha_k \, \phi(x,y,x_k,y_k)$$

$$B_r(x,y) = \sum_{l=1}^{L} \beta_{lr} \, \phi(x,y,x_l,y_l)$$

$$C_r(x,y) = \sum_{l=1}^{L} \gamma_{lr} \, \phi(x,y,x_l,y_l)$$

where

$$\phi(x,y,x_l,y_l) = \left[(x-x_l)^2 + (y-y_l)^2\right] \cdot \ln\left(\sqrt{(x-x_l)^2+(y-y_l)^2} - 1\right)$$

are the Green's function solution to the biharmonic equation

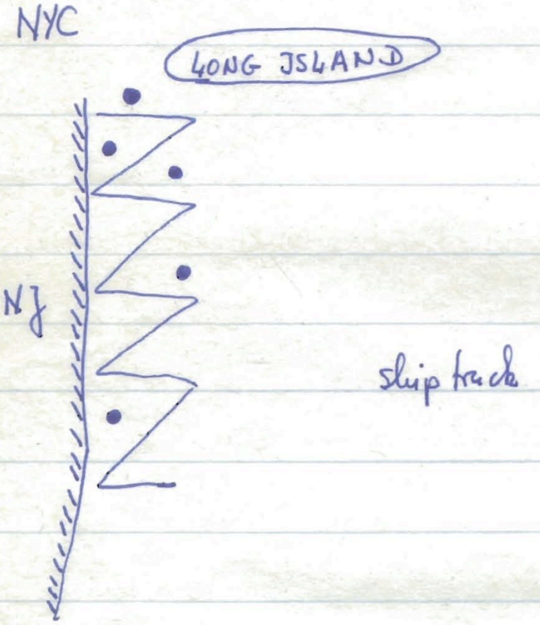
$$\nabla^4 A(\vec{x}) = \sum_{l=1}^{L} \alpha_l \, \delta(\vec{x}-\vec{x}_l)$$

$\vec{x} = (x,y)$
δ is δ fctn.

If $M = N$ is the number of data points to be fitted, then this corresponds to a biharmonic spline that fits the data exactly (but possibly oscillates and overshoots between pts. → Bryan's experience).

We have choose $M \ll N$ in order to get an overdetermined system to be fitted by least squares, ~~i.e.~~ i.e., do not fit the data exactly but removes some of the spurious oscillations/overshoot.

What, then, do the (x_ℓ, y_ℓ) $\ell = 1, 2, \dots L$ represent?



• location of "knots", i.e., (x_ℓ, y_ℓ) $\ell = 1, 2, \dots 5$ where we actually try to fit our model to parameters → the "smooth" biharmonic spline fctns. interpolated smoothly between these knot locations

example : $K = 0$ (no "mean" flow fitted)

$L = 1$ 1 knot

$M = 1$ 1 tidal constituent @ frequency ω

$$u(x,y,t) = a(x,y)\cos\omega t + b(x,y)\sin\omega t$$

$$= \alpha\,\phi(x,y)\cos\omega t + \beta\,\phi(x,y)\sin\omega t$$

where $\phi(x,y) = \cancel{(x-x)_\ell}\; x^2\left(\ln|\vec{x}| - 1\right)$ $\qquad x_i = (x_i - x_\ell,\ y_i - y_\ell)$

location of datum — location of knot

$$\varepsilon^2 = \sum_{i=1}^{N} \left\{ \frac{u_i - \phi(x-x_i, y-y_i)\left[\alpha \cos \omega t_i + \beta \sin \omega t_i\right]}{\sigma} \right\}^2$$

$$\frac{\partial \varepsilon^2}{\partial \alpha} = 2 \sum_{i=1}^{N} \left\{ \frac{u_i - \phi_i\left[\alpha \cos(\omega t_i) + \beta \sin(\omega t_i)\right]}{\sigma} \right\} \cdot (-)\phi_i \cos \omega t_i = 0$$

$$\frac{\partial \varepsilon^2}{\partial \beta} = 2 \sum_{i=1}^{N} \left\{ \frac{u_i - \phi_i\left[\alpha \cos(\omega t) + \beta \sin \omega t\right]}{\sigma} \right\} \cdot (-)\phi_i \sin \omega t_i = 0$$

$$\left[\frac{u\phi}{\sigma} \cos\right] - \alpha\left[\frac{\phi^2}{\sigma}\cos^2\right] - \beta\left[\frac{\phi^2}{\sigma}\cos \sin\right] = 0$$

$$\left[\frac{u\phi}{\sigma} \sin\right] - \alpha\left[\frac{\phi^2}{\sigma}\cos \sin\right] - \beta\left[\frac{\phi^2}{\sigma} \sin^2\right] = 0$$

or

$$\begin{pmatrix} \left[\frac{\phi^2}{\sigma}\cos^2\right] & \left[\frac{\phi^2}{\sigma}\cos \sin\right] \\ \left[\frac{\phi^2}{\sigma}\cos \sin\right] & \left[\frac{\phi^2}{\sigma}\sin^2\right] \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \left[\frac{u\phi}{\sigma}\cos\right] \\ \left[\frac{u\phi}{\sigma}\sin\right] \end{pmatrix}$$

σ may be different for different data sets, i.e., moorings vs. survey data have different uncertainties. 6/8/99

$$\underset{\sim}{\Phi} \qquad \vec{x} \qquad = \qquad \vec{D}$$

$$\phi = \phi(\underset{\substack{\uparrow \\ \text{data location}}}{x_i} - \underset{\substack{\uparrow \\ \text{knot location}}}{x_e},\ y_i - y_e) = \left[(x_i - x_e)^2 + (y_i - y_e)^2\right] \cdot \left[\ln\left(\sqrt{(x_i - x_e)^2 + (y_i - y_e)^2}\right) - 1\right]$$

example : $K = 0$

$L = 3$

$M = 2$

$$
\begin{array}{c|cccccccccccc|}
 & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{C1 \equiv \cos(\omega_1 t)}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{C2 = \cos(\omega_2 t)}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{S1 \equiv \sin(\omega_1 t)}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{S2 \equiv \sin(\omega_2 t)}} \\
 & \text{node 1} & \text{node 2} & \cancel{\text{node 3}}\ \text{node 3} & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1 & [C1\phi_1 \cdot C1\phi_1] & [C1\phi_2 \cdot C1\phi_1] & [C1\phi_3 \cdot C1\phi_1] & & & & [S1\phi_1 \cdot C1\phi_1] & & & & & \\
2 & [C1\phi_1 \cdot C2\phi_2] & & & & & & [S1\phi_1 \cdot C1\phi_2] & & & & & \\
3 & [C1\phi_1 \cdot C1\phi_3] & & & & & & & & & & & \\
4 & [C1\phi_1 \cdot C2\phi_1] & & & & & & & & & & & \\
5 & [C1\phi_1 \cdot C2\phi_2] & & & & & & & & & & & \\
6 & [C1\phi_1 \cdot C2\phi_3] & & & & & & & & & & & \\
7 & [C1\phi_1 \cdot S1\phi_1] & & & & & & & & & & & \\
8 & [C1\phi_1 \cdot S1\phi_2] & & & & & & & & & & & \\
9 & [C1\phi_1 \cdot S1\phi_3] & & & & & & & & & & & \\
10 & [C1\phi_1 \cdot S2\phi_1] & & & & & & & & & & & \\
11 & [C1\phi_1 \cdot S2\phi_2] & & & & & & & & & & & \\
12 & [C1\phi_1 \cdot S2\phi_3] & & & & & & & & & & &
\end{array}
\qquad
\vec{D} = \begin{pmatrix} \cancel{C1\phi_1\, u} \\ [C1\phi_1\, u] \\ [C2\phi_2\, u] \\ [C1\phi_3\, u] \\ [C2\phi_1\, u] \\ [C2\phi_2\, u] \\ [C2\phi_3\, u] \\ [S1\phi_1\, u] \\ [S1\phi_2\, u] \\ [S1\phi_3\, u] \\ [S2\phi_1\, u] \\ [S2\phi_2\, u] \\ [S2\phi_3\, u] \end{pmatrix}
$$

~~What about errors~~ !? $[\,\cdot\,] = \sum_{i=1}^{N} (\cdot)$ N is # of data used in the fit

end of class #18

What about errors ? Lots of handwaving as I do not want to get too deeply into multiple regression

$$\underset{\sim}{F}\,\vec{x} = \vec{D} \qquad\qquad D_j = \sum_{\nu=1}^{N} d_{j\nu}\cdot u_\nu = [d_j u]$$

Assume that the measurements u_ν are drawn from a normal distribution with a true variance $VAR(u_\nu) = \sigma^2$ about a true mean; the parameters

$$\vec{x} = \underset{\sim}{F}^{-1}\,\vec{D}$$

are also random and normally distributed with a variance

$$VAR(x_i) = F_{ii}^{-1}\cdot\sigma^2$$

because it results from a linear system and because we can interprete (no proof) the matrix $\underset{\sim}{F}^{-1}$ as the covariance matrix for the parameters $\vec{x}$, i.e.

$$COV(x_i, x_j) = \sigma^2\, F_{ij}^{-1}$$

$$\cancel{VAR(x_i, x_i) = \sigma^2\, F_{ii}^{-1}}$$

where

$$\sigma^2 \approx \varepsilon^2 / \underbrace{[N - (24\cdot M + K)]}_{\text{effective d.o.f. (degrees of freedom)}}$$

T

what's ε ?

$$\varepsilon^2 = \left(u_\nu - u(x_\nu, y_\nu, t_\nu)\right)^2$$

The ratio

$$x_i \Big/ \sqrt{F_{ii}^{-1}\,\sigma^2}$$

has a Students t-distribution with $N-(24M+K)$ degrees of freedom and we can estimate ~~95%~~ α (or any other) confidence limits (really hypothesis testing) as

$$\Delta x_i = t_{\alpha,\,d.o.f.} \cdot \sqrt{F_{ii}^{-1}\,\sigma^2} \qquad \text{for } \alpha = 0.05 \text{, i.e., 95\% confidence}$$

$$= t_{\alpha,\,d.o.f} \cdot \sigma \cdot \sqrt{F_{ii}^{-1}} \qquad d.o.f > 20$$

and $\qquad$ & $t_{\alpha,d.o.f} \approx 1.96$

$$\Delta u_\gamma = t_{\alpha,d.o.f} \cdot \sigma \cdot \sqrt{\varepsilon_\gamma^2} \;= t_{\alpha,dof} \cdot \frac{\varepsilon^2}{d.of} \quad \left[\;\sigma^2 = \frac{\varepsilon^2}{dof}\right.$$

$$\varepsilon_\gamma^2 = \left(u_\gamma - u(x_\gamma, y_\gamma, t_\gamma)\right)^2$$

references: – Brownlee, K.A. 1965. Statistical theory and methodology in science and engineering. J. Wiley and Sons. New York, NY.

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