1 Frequency Domain EOF → phase propagation of waves/oscillations
→ time lag problems

conventional (time domain) EOF detects standing oscillations only
"patterns" are either in or out of phase.

complex (frequency domain) EOF detects propagating and standing oscillations
"patterns" can move, i.e., time-space lagged correlations

I. brute force approach on set of series $\{x\} = \{x_1(t), x_2(t), \ldots, x_N(t)\}$

1. set up (complex) cross-spectral matrix

$$S_{ij}(f) = \frac{1}{T} E[X_i^*(f)\, X_j(f)]$$

set-up covariance matrix

$$R_{ij} = \langle x_i(t)\, x_j(t) \rangle$$

2. find eigenvalues of matrix $S_{ij}(f)$

→ $\lambda_i = \lambda_i(f)$ $i = 1, 2, \ldots N$

(power spectra)

as S_{ij} is Hermetian, ~~the eigen~~
the eigenvalues are real

find eigenvalues of matrix R_{ij}

→ λ_i $i = 1, 2, \ldots N$

(variance explained)

as R_{ij} is symetric
the eigenvalues are real

3. find eigenvectors of matrix S_{ij}
for each eigenvalue $\lambda_i(f)$

eigenvectors are now complex
different set of vectors at each frequency

find eigenvectors of matrix R_{ij}
for each eigenvalue λ_i

4. find the amplitude series
for each eigenvector

Michelson (1982)
JPO 694-703

??? $A_n(f_k) = \sum_{i=1}^{M} X_i(x_i, f_k)\, \phi_n(x_i)$

find the amplitude series

$$A_n(t_k) = \sum_{i=1}^{M} u_i(x_i, t_k)\, \phi_n(x_i)$$

Brute force approach to frequency domain EOF

A set of observations are

$$\vec{u}(t) = \{ u_1(t,x_1), u_2(t,x_2), \ldots, u_M(t,x_M) \} = \{ u_1(t), u_2(t), \ldots u_M(t) \}$$

An EOF analysis looks at

$$\underset{\sim}{R}_{pq}(\tau=0) = \langle u_p(t)\, u_q(t) \rangle$$

while FEOF analysis looks at

$$\underset{\sim}{R}_{pq}(\tau) = \langle u_p(t)\, u_q(t+\tau) \rangle$$

in the form

$$\underset{\sim}{S}_{pq}(f) = \int_{-\infty}^{\infty} \underset{\sim}{R}_{pq}(\tau)\, e^{-j 2\pi f \tau}\, d\tau \cong \frac{1}{T} U_p^*(f)\, U_q(f)$$

The EOF (FEOF) analysis tries to find the real (real) eigenvalues λ_i and corresponding real (complex) eigenvectors $\vec{\phi}_i$ of the real (complex) and symetric (Hermetian) matrix $\underset{\sim}{R}_{pq} = \underset{\sim}{R}_{qp}$ $\left(\underset{\sim}{S}_{pq} = \underset{\sim}{S}^*_{qp} \right)$, i.e.,

$$\underset{\sim}{R}_{pq} \cdot \vec{\phi}_i = \lambda_i \vec{\phi}_i \qquad \left(\underset{\sim}{S}_{pq} \cdot \vec{\phi}_i = \lambda_i \vec{\phi}_i \right) \quad i = 1,2,\ldots M$$

The real EOF amplitude functions are $\quad A_n(t) = \sum_{i=1}^{M} u_i(t)\, \phi_n(x_i)$

while the complex FEOF amplitudes are

$$A_n(t) = \sum_{i=1}^{M} u_i(t) \int_{-\infty}^{\infty} \phi_n(x_i, f)\, e^{j 2\pi f t}\, df$$

II. "Wide band" frequency domain EOF (Barnett (1983). Monthly Weather Review p. 756 – 773
also Horel (1984). J. Climate Appl. Meteor. 1660–1673)

- widely used, especially by meteorologists and climatologists
- essentially an average over all frequencies of the cross-spectral matrix is analysed; average can also be over a "wide" frequency band

A set of scalars (can be generalized to vectors → Barnett (1983)) are

$$\vec{u}(t) = \{u_1(t,x_1), u_2(t,x_2), \ldots, u_M(t,x_M)\} = \{u_1(t), u_2(t), \ldots, u_M(t)\}$$

and have a "harmonic" or "Fourier" representation

p – "location"
q for frequency

$$u_p(t) = \sum_{q=1}^{N} a_p \cos\left(2\pi \frac{q}{T} t\right) + b_p \sin\left(2\pi \frac{q}{T} t\right)$$

$$= \sum_{\omega} a_p(\omega) \cos \omega t + b_p(\omega) \sin(\omega t) \qquad \omega = \frac{2\pi}{T} i$$

In order to describe traveling (time lagged) feature we need a complex representation such as

$$U_p(t) = \sum_{\omega} c_p(\omega)\, e^{-j\omega t} \qquad c_p(\omega) = a_p(\omega) + i\, b(\omega)$$

$$= \sum_{\omega} \left[a_p(\omega) + j\, b_p(\omega)\right]\left[\cos \omega t - j \sin \omega t\right]$$

$$= \sum_{\omega} a_p(\omega) \cos \omega t + b_p(\omega) \sin \omega t + j\left(b_p(\omega) \cos \omega t - a_p(\omega) \sin \omega t\right)$$

$$U_p(t) = u_p(t) + j\hat{u}_p(t)$$

$$Re(U_p) = \sum_\omega a_p(\omega)\cos\omega t + b_p(\omega)\sin\omega t$$

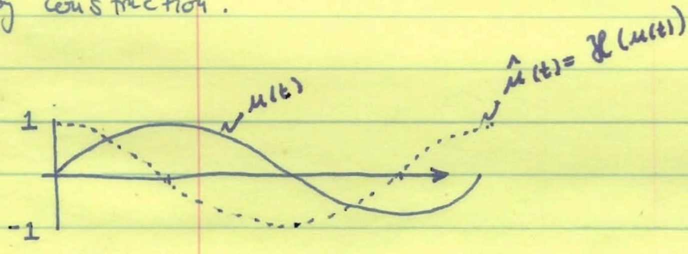
$$Im(U_p) = \sum_\omega b_p(\omega)\cos\omega t - a_p(\omega)\sin(\omega t)$$

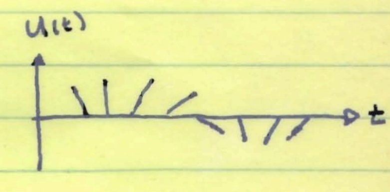
$$= \sum_\omega b_p(\omega)\sin(\omega t + \pi/2) + a_p(\omega)\cos(\omega t + \pi/2)$$

→ $\hat{u}_p(t)$ represents $u_p(t)$ phase shifted by $\pi/2$;
same Fourier coefficients !

This represents formally a Hilbert transform, i.e., u_p and $\hat{u}_p$ are Hilbert transform pairs. Much is known about them (similar to Fourier transforms) and their characteristics. They are orthogonal by construction.

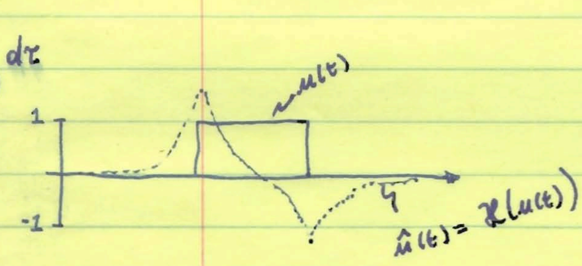
examples

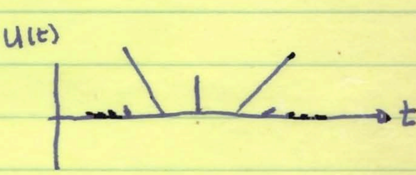




$$\hat{u}(t) = \mathcal{H}(u(t)) = \int_{-\infty}^{\infty} \frac{u(t')}{\pi(t-t')}\,d\tau$$

linear operator
Bendat + Piersol (1986)
last chapter





How to get the Hilbert transform in practice

(1) from ~~its~~ the Fourier coefficients (FFT) of the original data, i.e., get $a(\omega)$, $b(\omega)$
(2) design a "filter" that changes phase by $\pi/2$ but leaves amplitude intact

The complex covariance matrix

$$R_{pq} = \langle U_p^*(t)\, U_q(t) \rangle$$

represents an average of the cross-spectral matrix $\overset{S_{pq}(f)}{\vee}$ over the entire frequency domain because

$$R_{pq}(\tau=0) = \int_{-\infty}^{\infty} R_{pq}(\tau)\, \delta(\tau=0)\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{pq}(\tau) \int_{-\infty}^{\infty} 1 \cdot e^{-j\,2\pi f \tau}\, df\, d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{pq}(\tau)\, e^{-j\,2\pi f\tau}\, df\, d\tau$$

$$\underset{\substack{\uparrow \\ \text{handwaving}}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{pq}(\tau)\, e^{-j\,2\pi f\tau}\, d\tau\, df$$

$$= \int_{-\infty}^{\infty} S_{pq}(f)\, df$$

The eigenvalue problem we now write again as

$$R_{pq} \cdot \vec{\phi} = \lambda\, \vec{\phi}$$

where we get (as in the standard EOF analysis) from the eigenvectors $\vec{\phi}$

$$U_p(t, x_p) = \sum_{n=1}^{M} A_n(t) \cdot \phi_n^*(x_i) \qquad \sum_{n=1}^{M} \phi_p(x_n)\, \phi_q^*(x_n) = \delta_{pq}$$

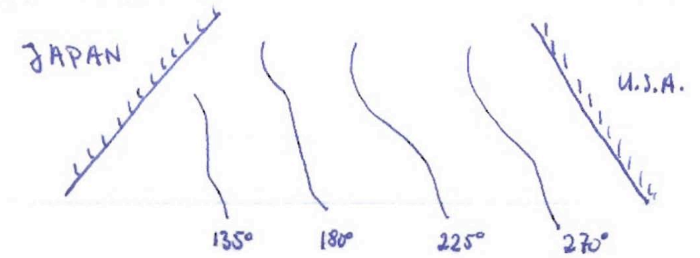
with

$$A_n(t) = \sum_{p=1}^{M} U_p(x_p, t)\, \phi_n(x_i) \qquad \frac{1}{K}\sum_{l=1}^{K} A_n(t_l)\, A_m^*(t_l) = \delta_{nm}\, \lambda_n$$

Because both eigenvectors and amplitude series are complex, we can now define / interpret those as 4 different functions such as

at each frequency
for each eigenvector

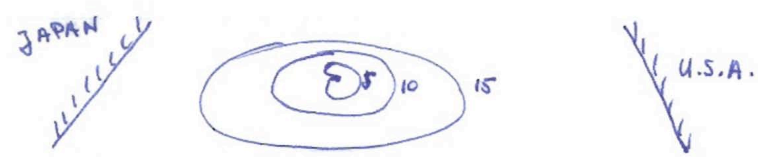
1. a spatial phase
at each EOF mode n: $\Theta_n (X_p) \equiv - \tan^{-1} \left[\mathrm{Im} (\phi_n (x_p)) / \mathrm{Re} (\phi_n (x_p)) \right]$



for a sinusoid
$u(x,t) = a \sin(kx - \omega_o t)$
there will be just 1 mode that goes through 2π over some distance L
i.e. $\Theta_1 = kx = \frac{2\pi}{L} x$

2. a spatial amplitude

$$B_n (X_p) = \sqrt{\phi_n (x_p) \, \phi_n^* (x_p)}$$

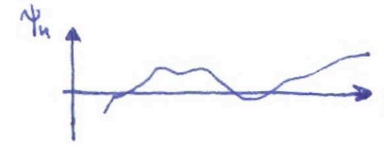


amplitude would be independent of x

3. a temporal phase

$$\Psi_n (t) = \tan^{-1} \left[\mathrm{Im} (A_n (t)) / \mathrm{Re} (A_n (t)) \right]$$

also indicate non-stationarity to some extend

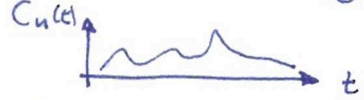


a pure sinusoid
$u(x,t) = a \sin(kx - \omega_o t)$
would give 

like complex demodulation ?

4. a temporal amplitude

$$C_n (t) = \sqrt{A_n (t) \, A_n^* (t)}$$



Summary EOF / FEOF / CEOF Analysis
Principal component analysis
Factor analysis
~~multip~~

- economy of description — data reduction; reveal redundancies
- orthogonalization — transform data into a set of
- quantification of variance distribution — mutually uncorrelated data
- applies to non-stationary data
- separates spatial from temporal variability
- FEOF / CEOF detect propagating "pattern" also
  EOF detect standing "pattern" only
- FEOF works best if signal is concentrated in a narrow frequency band
- CEOF performance improves by band-pass filtering data
- interpretation of EOF / CEOF / FEOF not always easy
  often need additional supporting data to
  physically interpret the statistical decomposition

- active area of research; mostly meteorological application
  (but then; oceanography always a few years / decades behind)