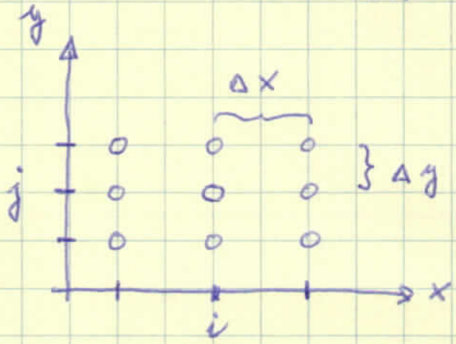
Start with General Mapping Tools

Gridding : Estimate values z from observations

z_i at locations (x_i, y_i):

$$z = z(x, y)$$

In two spatial dimensions find



$$z_{ij} = z\,(x = i \cdot \Delta x,\ y = j \cdot \Delta y)$$

Δx limits information content
Nyquist wavelength $\lambda_{N_x} = 1/2\Delta x$

Δy Nyquist wavelength $\lambda_{N_y} = 1/2\Delta y$

Three concerns:

1. Global properties of solution
2. Fit observation exactly or approximately
3. How to interpolate / extrapolate into poorly constrained region with few or no data?

Assumptions:

(a) Gridded function $z(x,y)$ is single-valued at (x,y)

(b) $z(x,y)$ is continuous in region

(c) $z(x,y)$ is positively auto-correlated over some length scale

Statistical Methods

minimize variance by selecting weights to data based on auto-correlation of data

→ Kriging
+ yield confidence limits for z_{ij}
− global properties not assured (high-order continuity, smoothness)

Integral Methods

minimize some global norm over some set of functions of the data

→ Minimal Curvature
+ assures solution with desired global property
− not easy to get confidence limits

2-D Minimum-Curvature

will become a smoothing spline which does not fit the data exactly

Use the Semi-Norm C

$$C = \iint \left(\nabla^2 z\right)^2 \, dx\, dy$$

that approximates total curvature of $z(x,y)$ for small $|\nabla z|$

Minimizing C gives the differential equation

$$\nabla^2(\nabla^2 z) = \sum_{i=1}^{N} \underbrace{f_i}_{\text{unknown weights}} \underbrace{\delta(x-x_i,\ y-y_i)}_{\text{loading at data locations}}$$

lengthy proof in Briggs (1974)

where $z_i = z_i(x_i, y_i)$ are observation $i = 1, 2 \ldots N$

The f_i must be chosen such that

$$z \longrightarrow z_i \quad \text{as } (x,y) \longrightarrow (x_i, y_i)$$

Boundary conditions are

$$\frac{\partial^2 z}{\partial n^2} = 0 \qquad \text{and} \qquad \frac{\partial}{\partial n}(\nabla^2 z) = 0 \qquad @ \text{ Edges}$$

and

$$\frac{\partial^2 z}{\partial x \partial y} = 0 \qquad @ \text{ Corners}$$

Note that $\left[\nabla^2(\nabla^2 z)\right] = \nabla^4 z = \sqrt{= \frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4}}$ is the biharmonic operator

Unique solutions exist with contineous second derivatives that are called <u>natural bicubic splines</u>

[analogy with elastic-plate flextures]

Find solution either by

(1) Finite Differences (Briggs, 1974) or

(2) Least-Square Matrix (Sandwell, 1987)

$$\nabla^4 z = \sum_{j=1}^{N} f_j \, \delta(\vec{x} - \vec{x}_j) \qquad \vec{x} = (x, y)$$

$$z(\vec{x}_i) = z_i$$

[see also p. 117, 118]

General solution is

$$z(\vec{x}) = \sum_{j=1}^{N} f_j \, \phi(\vec{x} - \vec{x}_j)$$

where for 2-dimensions $\phi(\vec{x}) = |\vec{x}|^2 (\ln |\vec{x}| - 1)$
is the biharmonic Green function.
We are left to find the unknown f_j from

$$z_i = \sum_{j=1}^{N} f_j \cdot \phi(\vec{x}_i - \vec{x}_j)$$

This is a linear set of N equations that can be written
as matrix form as

$$\vec{Z} = \underset{\sim}{\Phi} \cdot \vec{F}$$

where $\vec{Z} = (z_1, z_2, \dots z_N)$ are known data

$\underset{\sim}{\Phi}$ are known (Green) functions depending on data locations (Data Kernel)

$\vec{F}$ are the unknowns that complete the solution

The solution by inversion of matrix $\underset{\sim}{\Phi}$ fits all N data points exactly subject to the constraint of minimal curvature, however, solution may oscillate wildly between data locations.

1-D example (Sandwell, 1987)

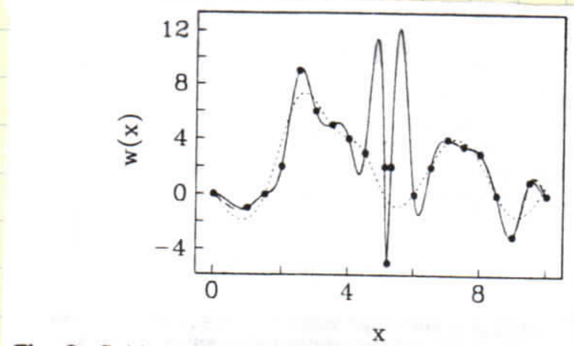


Fig. 2. Cubic spline interpolation (solid curve) of 21 data points (circles) where end slopes are zero. Biharmonic spline interpolation (dashed curve) using the one-dimensional Green function where end slopes are unconstrained. Biharmonic spline interpolation (dotted curve) where every other point force was omitted.

Two Solutions

1. Formulate the problem as one of least-square function fitting as done on p. 117-118 of notes

2. Consider "Splines in Tension"

"small displacement z of thin elastic plate of constant flexural rigidity D subject to a vertical normal stress q and constant <u>horizontal forces</u> per unit vertical length T_{xx}, T_{xy}, T_{yy} satisfy (Love, 1927)"

$$D \nabla^4 z - \underbrace{\left[T_{xx} \frac{\partial^2 z}{\partial x^2} + 2 T_{xy} \frac{\partial^2 z}{\partial x \partial y} + T_{yy} \frac{\partial^2 z}{\partial y^2} \right]}_{\text{new "tension"}} = q$$

Phyically, the f_i now represent the strength q/D of point loads on the elastic plate

Mathematically, the f_i are coefficients in a solution of a linear combination of Green's functions for plate flexture due to unit point loads.

Suppose $T_{xx} = T_{yy} = T$ and $T_{xy} = 0$

$$D \nabla^4 z - T \nabla^2 z = q$$

or

$$(1-t) \nabla^4 z - t \nabla^2 z = \sum_{i=1}^{N} f_i \, \delta(x - x_i, y - y_i)$$

and

t = constant tension parameter $\in [0, 1]$

$t = 0$ gives prior minimal curvature

$t = 1$ gives harmonic surface without any min/max except at data locations

Smith + Wessel (1990) "Gridding with continuous curvature splines in tension" Geophysics, 55

Tension relaxes the global minimal curvature constraint to find a solution with more local variations

$\longrightarrow$ emphasizes variance at shorter wavenumbers more

Practical Aspects

(A) remove the "mean" or "regional" field first

either a constant or a constant linear trend surface

$$z_k = a + b\hat{x}_k + c\hat{y}_k$$

[add this back when done gridding]

$\longrightarrow$ least-squares function fitting

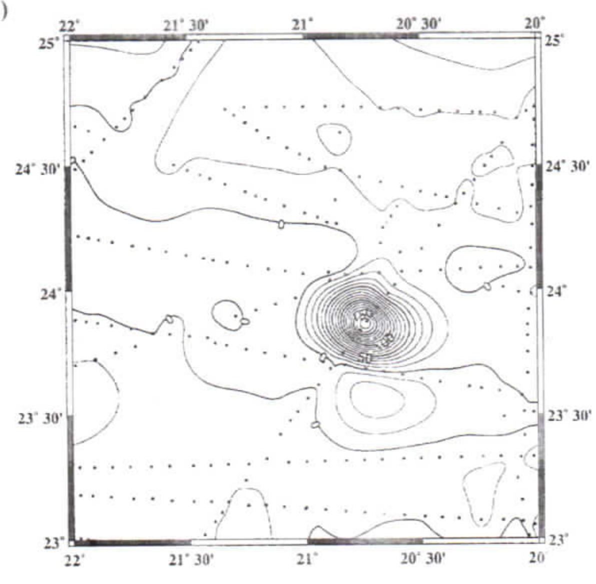
(B) The "Data Kernel" $\underset{\sim}{\Phi}$ or matrix-form solution to Green's function has intuitive appeal, but it is not always stable, because $\underset{\sim}{\Phi}$ often become sparse, nearly singular as the distance of data at $\vec{x}_i - \vec{x}_j$ become very large and small

$\longrightarrow$ finite differences used iteratively more stable and more efficient computationally

(C) Pre-Process Data via a spatial filter (block median) to reduce sensitivity of outliers or spurious data.

(D) Always show Data Location to judge contours and features as gridding can introduce artifacts.

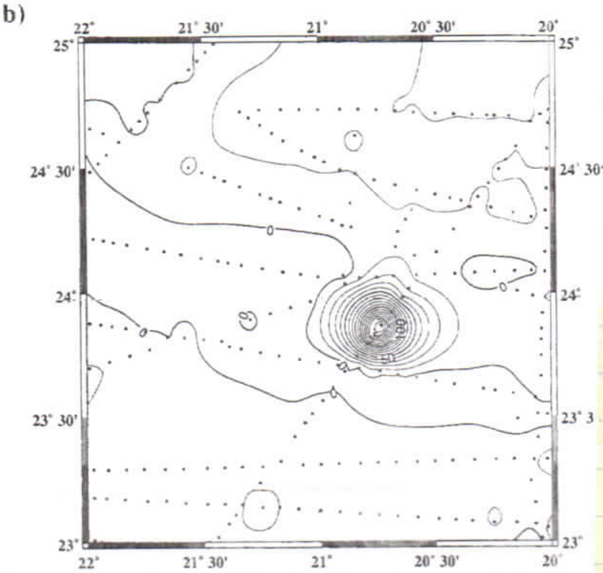
a)



No Tension

Dipole not supported by data

b)



Same Data with tension (0.3)

← Monopole supported by data

Smith + Wessel, 1990: Gridding with contineous curvature splines in tension, GEOPHYSICS, 55, 293-305.