

(12.5) Correlation Structures of Non-stationary Data

Consider a pair of non-stationary processes $\{x(t)\}$ and $\{y(t)\}$. The mean values at any fixed time t are

$$\mu_x(t) = E[x(t)] \quad \text{and} \quad \mu_y(t) = E[y(t)]$$

The correlation functions at any pair of fixed times t_1 and t_2 are defined by their expected values

$$R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$$

$$R_{yy}(t_1, t_2) =$$

$$R_{xy}(t_1, t_2) = E[x(t_1)y(t_2)]$$

} for stationary data only $(t_2 - t_1) \neq \tau$ mattered.

It follows that

$$R_{xx}(t_1, t_2) = R_{xx}(t_2, t_1)$$

$$R_{yy}(t_1, t_2) = R_{yy}(t_2, t_1)$$

$$R_{xy}(t_1, t_2) = R_{xy}(t_2, t_1)$$

and

$$|R_{xy}(t_1, t_2)|^2 \leq R_{xx}(t_1, t_2) \cdot R_{yy}(t_1, t_2)$$

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As before we estimate these functions by using N samples of $x_i(t)$ and compute an ensemble average

$$\hat{R}_{xx}(t_1, t_2) = \frac{1}{N} \sum_{i=1}^N x_i(t_1) \cdot x_i(t_2)$$

or rewrite with

$$t_1 = t \quad \text{and} \quad t_2 = t - \tau \quad (\text{or } \tau = t - t_2 = t_1 - t_2)$$

$$\hat{R}_{xx}(t, t - \tau) = \frac{1}{N} \sum_{i=1}^N x_i(t) x_i(t - \tau)$$

or rewrite with

$$t_1 = t - \tau/2 \quad t_2 = t + \tau/2$$

then

$$\tau = t_2 - t_1$$

$$t = (t_1 + t_2) / 2$$

time difference
lag

center time between t_1 and t_2

And

$$\begin{aligned} R_{xy}(t_1, t_2) &= R_{xy}(t - \tau/2, t + \tau/2) \\ &= E[x(t - \tau/2) \cdot y(t + \tau/2)] \\ &= R_{xy}(\tau, t) \end{aligned}$$

For a zero lag $\tau = 0$

$$R_{xx}(\tau=0, t) = E[x^2(t)] = \sigma_x^2 \quad \text{Variance}$$

$$R_{yy}(\tau=0, t)$$

$$R_{xy}(\tau=0, t) = E[x(t)y(t)] = \sigma_{xy} \quad \text{Co-Variance}$$

And symmetries are

$$R_{xx}(\tau, t) = R_{xx}(-\tau, t)$$

$$R_{yy}(\tau, t) =$$

$$R_{xy}(\tau, t) = R_{yx}(-\tau, t)$$

If we form an average over time, we get

$$\overline{R}_{xy}(\tau) = \lim_{T \rightarrow \infty} \int_0^T R_{xy}(\tau, t) dt$$

which is a real-valued, even function of τ representing the usual cross-correlation of stationary random data