ERGODIC processes are stationary processes for which a time average from a single realisation can replace an ensemble average.

$$\text{Mean} \quad \mu_x(t_1) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i(t_1) = \mu_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt$$

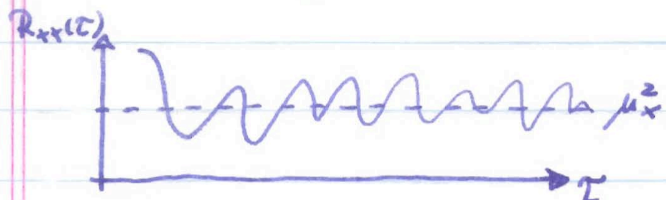
$$\text{Auto-correlation} \quad R_{xx}(t_1,\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i(t_1)\,x_i(t_1+\tau) = R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,x(t+\tau)\,dt$$

stationarity   ergodicity

emphasise the many assumptions !

A sufficient but not necessary condition for ergodicity is

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \left| R_{xx}(\tau) - \mu_x^2 \right| d\tau = 0$$



proof elsewhere
Yaglom (1987)

Because of all these assumptions it is always good practice to write/talk about "estimates" of properties of a physical/biological system. An "estimate", however, implies both

- probability of occurence of an event
- uncertainty of the "estimate"; relate to probability

tests for stationarity?

Recapture where I was:

Definition

Ensemble

$\left\{ \begin{array}{l} \text{Stationarity} \\ \text{Ergodicity} \end{array} \right\}$ random processes

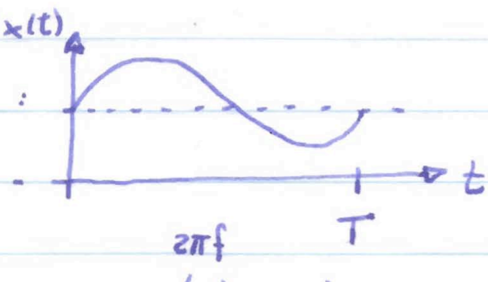
Also, so far everything has been written down for continuous, infinite long records, however, all measurements are both discontinuous and of finite length → possible biases and errors
→ statistical estimates, never exact values

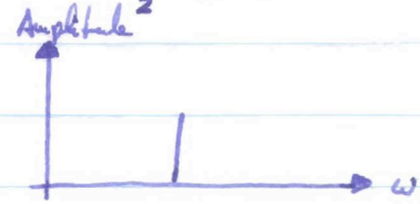
Will return later ← RANDOM (STOCHASTIC) PROCESSES

2. DETERMINISTIC PROCESSES

(a) periodic, i.e., $x(t) = x(t+T)$ of period T

example :





$$x(t) = \cancel{X}_0 \cos(\overset{2\pi f}{\cancel{\omega}}\, t + \phi)$$

$$= X_{01} \cos \underset{2\pi f}{\cancel{\omega}} t + x_{02} \sin \overset{2\pi f}{\cancel{\omega}} t$$

(3) complex periodic $x(t) = x(t + nT)$ $\quad n = 1, 2, 3, \ldots$

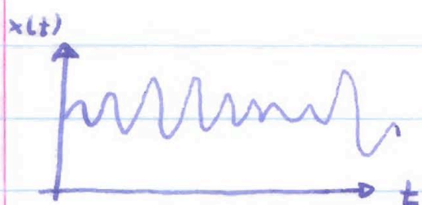
such data can always be written as a $\boxed{\text{Fourier Series}}$

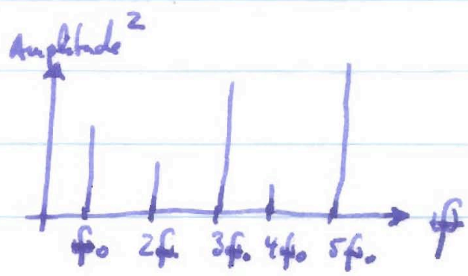
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \cdot 2\pi f_0\, t) + b_n \sin(n \cdot 2\pi f_0 \cdot t)$$

where $f_0 = \frac{1}{T}$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(2\pi f_0\, n\, t)\, dt \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n \cdot 2\pi f_0 \cdot t)\, dt \qquad n = 1, 2, 3, \ldots$$





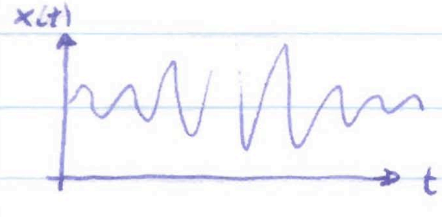
$\Delta f = f_0 = \frac{1}{T}$

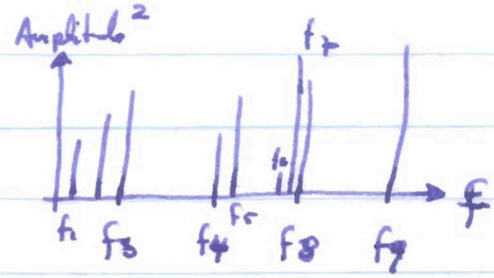
(c) almost periodic (~~but~~ nonperiodic)

f_i are not rational numbers & $T \to \infty$

$$x(t) = \sum_{i=1}^{\infty} x_i \cos(2\pi f_i t)$$

~~discrete~~ set of incommensurable periods



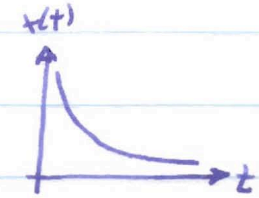


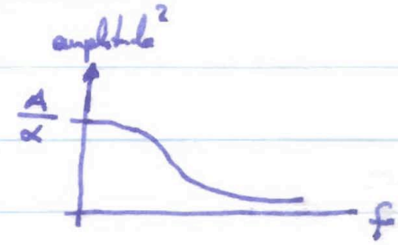
$\boxed{\text{DISCRETE SPECTRA}}$

Tides are of this form?

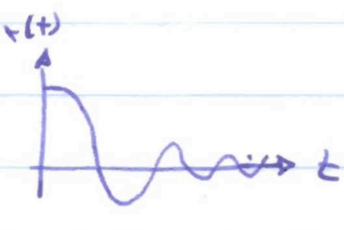
(d) transient data

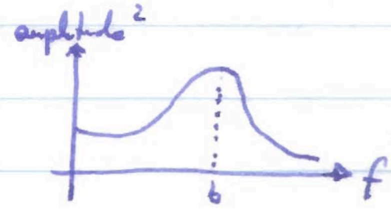
$$x(t) = \begin{cases} A e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



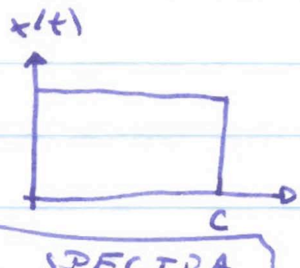


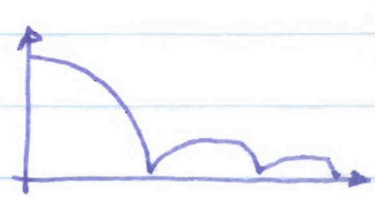
$$x(t) = \begin{cases} A e^{-\alpha t} \cos bt & t > 0 \\ 0 & t < 0 \end{cases}$$





$$x(t) = \begin{cases} A & c > t \geq 0 \\ 0 & c < t < 0 \end{cases}$$





$\boxed{\text{CONTINEOUS SPECTRA}}$

$\Longrightarrow$ Fourier Transforms

$\Longrightarrow$ applications?

end class #2/99

CLASS #2 class #3/99

Fourier Transform

In 1807 Joseph Fourier anounced that ANY periodic function $x(t) = x(t+T)$ could be represented by the form

$$x(t) = \sum_{i=0}^{\infty} a_i \cos\left(i \frac{2\pi}{T} t\right) + b_i \sin\left(i \frac{2\pi}{T} t\right)$$

$$= a_0 + \sum_{i=1}^{\infty} a_i \cos(\ ) + b_i \sin(\ )$$

APPLIES EVEN TO DISCONTINEOUS FUNCTIONS!

orthogonal set of base functions to span a function space

$$\langle \sin(nt) \cdot \sin(mt) \rangle = \delta_{nm}$$
$$\langle \cos(nt) \cdot \cos(mt) \rangle = \delta_{nm}$$
$$\langle \cos(nt) \cdot \sin(mt) \rangle = 0$$
$$\langle \cdot \rangle \equiv \frac{1}{T} \int \cdot \, dt$$

Louis Lagrange doubted this theorem

Riemann and Dirichlet found that some mild restrictions are needed

$$\int_{-T/2}^{+T/2} \cos\left(m \frac{2\pi}{T} t\right) \cdot \cos\left(n \frac{2\pi}{T} t\right) dt = \begin{cases} 0 & n \neq m \\ T/2 & m = n \geq 1 \\ T & m = n = 0 \quad \text{to } a_0 \end{cases}$$

$$\int_{-T/2}^{+T/2} \sin\left(m \frac{2\pi}{T} t\right) \cdot \sin\left(n \frac{2\pi}{T} t\right) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \geq 1 \end{cases}$$

$$\int_{-T/2}^{T/2} \sin\left(m \frac{2\pi}{T} t\right) \cdot \cos\left(n \frac{2\pi}{T} t\right) dt = 0$$

$$\int_{-T/2}^{T/2} x(t) \sin\left(n \frac{2\pi}{T} t\right) dt = b_n \cdot \frac{T}{2}$$

x(t)

$$\int_{-T/2}^{T/2} x(t)\, dt = \cancel{2} a_0 T \quad ; \quad \int_{-T/2}^{T/2} x(t) \cos\left(m \frac{2\pi}{T} t\right) dt = a_m \cdot \frac{T}{2}$$