Chapter 4 Sampling in Frequency and Time

4.1 Introduction

As a major application of using Fourier transforms of generalized functions, we consider the problems of Fourier series and of representing a continuous waveform by uniformly sampling it in time. Fourier series are used in the analysis of periodic waveforms while sampling theory is important in computer-aided processing of sampled physical data (other applications include compact disc players, digital audio etc.) Rather surprisingly, these topics are related and the theory is based on finding the Fourier transform of a train of delta functions.

4.2 Fourier transform of a train of delta functions

4.2.1 A train of $2N+1$ delta functions

Consider the generalized function consisting of $2N+1$ equally-spaced delta functions separated by time T

$$f_N(t) = \sum_{k=-N}^{N} \delta(t - kT) \tag{4.1}$$

The Fourier transform is found directly via linearity and the result for the Fourier transform of a single delta function

$$F_N(\nu) = \sum_{k=-N}^{N} \exp(-\mathrm{j}2\pi\nu kT) \tag{4.2}$$

$$= \frac{\sin\left[2\pi\left(N+\frac{1}{2}\right)\nu T\right]}{\sin\left(\pi\nu T\right)} \tag{4.3}$$

where we have summed the geometric series and written the complex exponential in trigonometric form. We note that

1. This is a "periodic" function (in ν) with period $1/T$
2. The major peaks are at k/T for integer k and the heights of the peaks are $2N+1$.
3. The zeros of the function are at $\nu = k/[(2N+1)T]$ where k is any integer not divisible by $2N+1$.

4.2.2 An infinite train of delta functions

Let us now consider what happens as $N \to \infty$. The sequence $f_N(t)$ tends in the distributional sense to

$$f(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{4.4}$$

You should check that this distributional limit does in fact hold by considering the action on a test function in $\mathcal{D}'$. Note that the definition of $\mathcal{D}'$ ensures that all results are well-defined and never infinite.

The Fourier transform of $f(t)$ is then the distributional limit of $F_N(\nu)$. Since F_N is periodic with period $1/T$ for every value of N, this periodicity is shared by the limit F. It thus suffices to consider a single period from $-1/(2T)$ to $1/(2T)$.

We look at the action of $F_N(\nu)$ on a test function $\Phi(\nu)$ in the interval $-1/(2T)$ to $1/(2T)$, i.e.,

$$\int_{-1/(2T)}^{1/(2T)} \frac{\sin\left[2\pi\left(N+\frac{1}{2}\right)\nu T\right]}{\sin\left(\pi\nu T\right)} \Phi(\nu)\,\mathrm{d}\nu \tag{4.5}$$

The denominator is nonzero except at $\nu = 0$. Thus as $N \to \infty$, the Riemann-Lebesgue lemma ensures that the contribution to the integral vanishes except within a small interval $(-\delta, \delta)$ around $\nu = 0$. Since the test function is assumed to be differentiable at zero, there is a neighbourhood in which its value is not appreciably different from $\Phi(0)$. The integral may be approximated by

$$\Phi(0) \lim_{N\to\infty} \int_{-\delta}^{\delta} \frac{\sin\left[2\pi\left(N+\frac{1}{2}\right)\nu T\right]}{\sin\left(\pi\nu T\right)} = \Phi(0) \lim_{N\to\infty} \frac{1}{(2N+1)\,\pi T} \int_{-(2N+1)\pi T\delta}^{(2N+1)\pi T\delta} \frac{\sin u}{\sin\left[u/\left(2N+1\right)\right]}\,\mathrm{d}u \tag{4.6}$$

As N becomes large, we may approximate $\sin[u/(2N+1)]$ by $u/(2N+1)$. Since $\int_{-\infty}^{\infty} \sin(u)/u\,\mathrm{d}u = \pi$, the result is $\Phi(0)/T$.

Thus within the interval $-1/(2T)$ to $1/(2T)$, $F_N(\nu) \to \delta(\nu)/T$. Using the periodicity of $F_N(\nu)$, we obtain the transform pair

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) \leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right) \tag{4.7}$$

Thus the Fourier transform of an infinite train of delta functions is also an infinite train of delta functions. The spacing in ν space is the reciprocal of the spacing in t space.

4.3 The Fourier transform of a periodic signal

Consider a function $f_p(t)$ which is periodic with period T. Define $f(t)$ to be a single period of $f_p(t)$ and which is zero outside the range $[-T/2, T/2]$, i.e.,

$$f(t) = \begin{cases} f_p(t) & \text{if } -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

We see that $f_p(t)$ may be regarded as the convolution of f with an infinite train of delta functions

$$f_p(t) = \sum_{k=-\infty}^{\infty} f(t-kT) = (f * h)(t) \tag{4.9}$$

where

$$h(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \tag{4.10}$$

Taking the Fourier transform and using the convolution theorem

$$F_p(\nu) = F(\nu)H(\nu) = \frac{1}{T}F(\nu)\sum_{k=-\infty}^{\infty}\delta\left(\nu - \frac{k}{T}\right) \tag{4.11}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}F\left(\frac{k}{T}\right)\delta\left(\nu - \frac{k}{T}\right) \tag{4.12}$$

Thus the spectrum of $f_p(t)$ consists of discrete components at multiples of $1/T$. The component of the periodic signal at frequency $1/T$ is called the **fundamental** and that at frequency k/T is called the k'th **harmonic**. We see that this harmonic relationship is a consequence of the periodicity of the original signal.

Let us now consider the form of the inverse Fourier transform which recovers $f_p(t)$ from $F_p(\nu)$:

$$f_p(t) = \int_{-\infty}^{\infty} F_p(\nu)\exp(\mathrm{j}2\pi\nu t)\,\mathrm{d}\nu \tag{4.13}$$

$$= \int_{-\infty}^{\infty}\frac{1}{T}\sum_{k=-\infty}^{\infty}F\left(\frac{k}{T}\right)\delta\left(\nu - \frac{k}{T}\right)\exp(\mathrm{j}2\pi\nu t)\,\mathrm{d}\nu \tag{4.14}$$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{T}F\left(\frac{k}{T}\right)\exp\left(\mathrm{j}2\pi\frac{k}{T}t\right) \tag{4.15}$$

$$= \sum_{k=-\infty}^{\infty}c_k\exp\left(\mathrm{j}2\pi\frac{k}{T}t\right) \tag{4.16}$$

where

$$c_k = \frac{1}{T}F\left(\frac{k}{T}\right) \tag{4.17}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp\left(-\mathrm{j}2\pi\frac{k}{T}t\right)\,\mathrm{d}t \tag{4.18}$$

where we have used the fact that $f(t)$ vanishes outside $[-T/2, T/2]$.

We recognize (4.16) as a Fourier series and (4.18) as the equations for the coefficients. By using the theory of generalized functions, we see that Fourier series can be recovered naturally from the theory of Fourier transforms.

Example: Find the Fourier series for $f_p(t)$ which is periodic of period T and which is defined within $[-T/2, T/2]$ by

$$f_p(t) = 1 - \frac{2\,|t|}{T} \quad \text{if } -\frac{T}{2} < t < \frac{T}{2} \tag{4.19}$$

The function $f(t)$ is defined by

$$f(t) = \begin{cases} 1 - \frac{2|t|}{T} & \text{if } -\frac{T}{2} < t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

The Fourier transform of $f(t)$ is

$$F(\nu) = \frac{T}{2}\,\mathrm{sinc}^2\left(\frac{\nu T}{2}\right) \tag{4.21}$$

The coefficients of the Fourier series are thus

$$c_k = \frac{1}{T} F\left(\frac{k}{T}\right) \tag{4.22}$$

$$= \frac{1}{2}\,\mathrm{sinc}^2\left(\frac{k}{2}\right) \tag{4.23}$$

$$= \begin{cases} \frac{1}{2} & \text{if } k = 0 \\ \dfrac{2}{\pi^2 k^2} & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{4.24}$$

The Fourier series is thus

$$f_p(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 (2n-1)^2} \cos\left(\frac{2\pi\,(2n-1)\,t}{T}\right) \tag{4.25}$$

Note that if we set $t = 0$, we find that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8} \tag{4.26}$$

Exercises:

1. Use this to show that
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6} \tag{4.27}$$
This is the value of the Riemann-zeta function at 2, denoted by $\zeta(2)$.

2. Find the Fourier series of the following functions which are periodic of period T
$$g_p(t) = \frac{t}{T} \quad \text{if } 0 < t < T \tag{4.28}$$
$$h_p(t) = \begin{cases} 1 & \text{if } |t| < T/4 \\ 0 & \text{if } T/4 < |t| < T/2 \end{cases} \tag{4.29}$$

4.4 Sampling in time

Often a function is known only by its values at particular (usually equally spaced) times. This is the case when we regularly sample a continuous-time waveform (e.g. digitized music). Mathematically we may represent this process of sampling at equally spaced points in time as multiplying the continuous-time function $f(t)$ by a train of delta functions to give a new (generalized) function $f_s(t)$.

$$f_s(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t - kT) \tag{4.30}$$

The second form shows explicitly that the values of f between sampling instants do not affect $f_s(t)$.

Corresponding to this product in the time domain, the spectrum $F(\nu)$ of $f(t)$ is convolved by the spectrum of the train of delta functions. Using the transform pair

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right) \tag{4.31}$$

we see that the Fourier transform $F_s(\nu)$ of $f_s(t)$ is the convolution

$$F_s(\nu) = F(\nu) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\nu - \frac{k}{T}\right) \tag{4.32}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} F\left(\nu - \frac{k}{T}\right) \tag{4.33}$$

The resulting spectrum now spans an infinite range of frequencies with repetitions of the original spectrum every $\nu_s = 1/T$ apart.

If the original spectrum of $f(t)$ is band-limited to $\pm\nu_c$ and if $\nu_c < \frac{1}{2}\nu_s$, the repetitions of the spectra will not overlap and we can recover the original spectrum $F(\nu)$ from $F_s(\nu)$.

On the other hand, if if $\nu_c > \frac{1}{2}\nu_s$, the repetitions of the spectra will overlap and the original spectrum is lost.

This tells us how rapidly we need to sample a waveform in order to allow the continuous-time waveform from the samples. These considerations are made precise in the following section.

4.5 The sampling theorem

If a function $f(t)$ is **band limited** so that its Fourier transform $F(\nu)$ vanishes for $|\nu| \geq \nu_c$, then $f(t)$ can be completely reconstructed from its values sampled at intervals of T provided that $\nu_s = T^{-1} \geq 2\nu_c$.

Proof: The set of samples of $f(t)$ taken with a sampling interval of T can be represented by the function $f_s(t)$ as defined by (4.30). The spectrum $F_s(\nu)$ is given by (4.33). If $T^{-1} \geq 2\nu_c$, the copies of the spectrum do not overlap and it is possible to recover the original spectrum $F(\nu)$ from $F_s(\nu)$ via

$$F(\nu) = TF_s(\nu)\Pi(\nu T) \tag{4.34}$$

By calculating the inverse Fourier transform of this relationship, we can recover $f(t)$ from $f_s(t)$. Using the transform pair

$$\operatorname{sinc}\left(\frac{t}{T}\right) \leftrightarrow T\Pi(\nu T) \tag{4.35}$$

we find that

$$f(t) = f_s(t) * \operatorname{sinc}\left(\frac{t}{T}\right) \tag{4.36}$$

$$= \int_{-\infty}^{\infty} f_s(\tau) \operatorname{sinc}\left(\frac{t-\tau}{T}\right) \mathrm{d}\tau \tag{4.37}$$

Substituting the expression for $f_s(\tau)$ as a sum of delta functions,

$$f(t) = \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} f\left(kT\right)\, \delta\left(\tau - kT\right) \right) \operatorname{sinc}\left(\frac{t-\tau}{T} \right)\, \mathrm{d}\tau \tag{4.38}$$

$$= \sum_{k=-\infty}^{\infty} f(kT) \operatorname{sinc}\left(\frac{t-kT}{T} \right) \tag{4.39}$$

This is the desired formula which expresses $f(t)$ for any time t in terms of the values of f at the sample instants kT.

1. The reconstructed value of $f(t)$ is a weighted mean of **all** the sample values $f(kT)$. The weight associated with those samples which are far away from t decreases according to the sinc function. If t happens to be a multiple of T the sinc function has a weight of one at that sample point and zero for all other samples.

2. In the frequency domain, the reconstruction of $f(t)$ from $f_s(t)$ involves an ideal low-pass filtering which removes all frequencies above half the sampling frequency. The impulse response of such an ideal low-pass filter is a sinc function, indicating that the filter is non-causal. In practice, we can only do this filtering approximately. Compact disc players use a combination of digital and analogue techniques (oversampling by digital interpolation followed by analogue postfiltering) to do this reconstruction.

3. The highest frequency which can be reconstructed from its samples using (4.39) is $\frac{1}{2}\nu_s = 1/(2T)$. This is called the **Nyquist frequency**. When digitizing a signal, all frequencies outside the Nyquist frequency band $(-\frac{1}{2}\nu_s, \frac{1}{2}\nu_s)$ **must** be removed (using an analogue filter) before the digitization. If this is not done, any frequency components ν outside the Nyquist frequency band (i.e., $|\nu| > \frac{1}{2}\nu_s$) will reappear in the reconstruction at a frequency $\nu + m\nu_s$ where m is an integer such that $|\nu + m\nu_s| \le \frac{1}{2}\nu_s$. This phenomenon is called **aliasing**.

4. Aliasing can sometimes be usefully employed to make one frequency appear like another. It is the principle used by the stroboscope in which a system is sampled at such a rate that its motion is apparently slowed down or "frozen" (e.g., the angular frequency of a rotating shaft is aliased to zero frequency) by the sampling.

5. Equation (4.39) can be interpreted as saying that a band limited signal can be written as a linear combination of sinc functions. Thus the (countable) family of functions

$$\left\{ \operatorname{sinc}\left(\frac{t-kT}{T} \right) \right\}_{k=-\infty}^{\infty} \tag{4.40}$$

may be regarded as a set of basis functions for the space of bandlimited functions whose spectra vanish outside $(-1/(2T), 1/(2T))$.

Exercise:

1. Show that these basis functions are orthogonal. In fact,

$$\int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t-mT}{T} \right) \operatorname{sinc}\left(\frac{t-nT}{T} \right)\, \mathrm{d}t = T\, \delta_{mn} \tag{4.41}$$

Hence show that for bandlimited functions $f(t)$ and $g(t)$ such that $F(\nu) = 0$ and $G(\nu) = 0$ for $|\nu| \geq 1/(2T)$,

$$\int_{-\infty}^{\infty} f^*(t)g(t)\,\mathrm{d}t = T \sum_{k=-\infty}^{\infty} f^*(kT)g(kT) \tag{4.42}$$

assuming that the appropriate integrals and sums converge. Thus we can calculate inner products of bandlimited signals in terms of their sampled values.

2. Instead of ideal reconstruction of a sampled signal by means of convolution with a sinc function, reconstruction using a **zero-order hold** is commonly used. This involves holding the value of the signal constant at the value of the last sample until the next sample arrives so that if $f(kT)$ are the sample values, the reconstruction using zero-order hold is

$$f_{ZOH}(t) = f(kT) \quad \text{for } kT \leq t < (k+1)\,T \tag{4.43}$$

Show that $f_{ZOH}(t)$ is related to $f_s(t)$ via a convolutional relationship and hence calculate the spectrum $F_{ZOH}(\nu)$ of the reconstruction.

4.6 Bernstein's Theorem

If $f(t)$ is a **bounded**, **bandlimited** function, i.e.,

1. There exists $f_{\max}$ such that $|f(t)| \leq f_{\max}$ for all t, and
2. There exists $\nu_{\max}$ such that the Fourier transform $F(\nu)$ of $f(t)$ vanishes for $|\nu| > \nu_{\max}$.

then

$$|f'(t)| \leq 2\pi\nu_{\max} f_{\max} \qquad \text{for all } t \tag{4.44}$$

This is a quantitative expression of the idea that bandlimited functions cannot change too quickly. The function $\cos(2\pi\nu_{\max}t + \phi)$ shows that the bound is achieved.

Proof:

The sampling theorem shows that for $T = 1/(2\nu_{\max})$ and for any τ we can write

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT+\tau)\,\mathrm{sinc}\left(\frac{t-\tau-kT}{T}\right) \tag{4.45}$$

(if this is not obvious, consider the bandlimited function $g(t) = f(t+\tau)$ and write Eq. 4.39 for this function).

The derivative of this is

$$f'(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} f(kT+\tau)\,\mathrm{sinc}'\left(\frac{t-\tau-kT}{T}\right) \tag{4.46}$$

where $\mathrm{sinc}'(x) = [\pi x \cos(\pi x) - \sin(\pi x)]/(\pi x^2)$. Notice how we have expressed the **derivative** of f in terms of equally-spaced **samples** of f itself.

We now choose $\tau = t + \frac{1}{2}T$. With this choice

$$f'(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} f\left(t + kT + \tfrac{1}{2}T\right) \operatorname{sinc}'\left(-k - \tfrac{1}{2}\right) \tag{4.47}$$

and so

$$|f'(t)| \le \frac{1}{T} \sum_{k=-\infty}^{\infty} \left|f\left(t + kT + \tfrac{1}{2}T\right)\right| \left|\operatorname{sinc}'\left(-k - \tfrac{1}{2}\right)\right| \tag{4.48}$$

$$\le \frac{f_{\max}}{T} \sum_{k=-\infty}^{\infty} \left|\operatorname{sinc}'\left(-k - \tfrac{1}{2}\right)\right| \tag{4.49}$$

But

$$\sum_{k=-\infty}^{\infty} \left|\operatorname{sinc}'\left(-k - \tfrac{1}{2}\right)\right| = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \left(k + \tfrac{1}{2}\right)^{-2} = \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} = \pi \tag{4.50}$$

where we have used the sum in Eq. (4.26).

Hence

$$|f'(t)| \le \frac{\pi}{T} f_{\max} = 2\pi\nu_{\max} f_{\max} \tag{4.51}$$

4.7 The discrete-time Fourier transform

We have seen in (4.33) that the Fourier transform $F_s(\nu)$ of a sampled function

$$f_s(t) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t - kT) \tag{4.52}$$

can be written as a convolution of $F(\nu)$ with a train of delta functions. We may alternatively calculate the Fourier transform of $f_s(t)$ directly by substituting it into the definition. This yields

$$F_s(\nu) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(kT)\delta(t - kT) \exp(-\mathrm{j}2\pi\nu t)\,\mathrm{d}t \tag{4.53}$$

$$= \sum_{k=-\infty}^{\infty} f(kT) \exp(-\mathrm{j}2\pi\nu kT) \tag{4.54}$$

Instead of regarding $f(kT)$ as a sampled version of some continuous-time function $f(t)$, we may think of them simply as a sequence of numbers. This leads to the definition

Definition: The **discrete-time Fourier transform** (DTFT) maps a sequence of numbers $\{x[k]\}$ for integer k into a continuous function $X(\Omega)$ defined by

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] \exp(-\mathrm{j}\Omega k) \tag{4.55}$$

The function $X(\Omega)$ is periodic with period 2π. The inverse discrete-time Fourier transform relationship is

$$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \exp(\mathrm{j}\Omega k)\,\mathrm{d}\Omega \tag{4.56}$$

Proof: Substituting the expression for $X(\Omega)$ into (4.56) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} x[n]\exp(-\mathrm{j}\Omega n)\exp(\mathrm{j}\Omega k)\,\mathrm{d}\Omega = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x[n]\int_{-\pi}^{\pi}\exp[\mathrm{j}\Omega(k-n)]\,\mathrm{d}\Omega \tag{4.57}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} 2\pi\,\mathrm{sinc}(k-n)x[n] \tag{4.58}$$

$$= x[k] \tag{4.59}$$

Exercises:

1. If the sequence $x[k]$ does happen to come from sampling a bandlimited signal, i.e., $x[k] = f(kT)$, show that we can recover the inverse DTFT relationship (4.56) from the usual inverse Fourier transform together with the relationship (4.34).

2. Show that the DTFT of the sequence

$$x[k] = \exp(-\alpha|k|) \tag{4.60}$$

is

$$X(\Omega) = \frac{1-e^{-2\alpha}}{1-2e^{-\alpha}\cos\Omega + e^{-2\alpha}} \tag{4.61}$$

3. Use the inverse DTFT relationship (4.56) to recover $x[k]$ from (4.61).