Please review carefully theoretical concepts of how the the lagged auto-correlation function \( R(\tau) \) of an observable \( x(t) \) and its Fourier transform \( X(f) \) relate to to the Fourier transforms of said auto-correlation function. It will simplify both your coding and the calculations substantially, e.g., \( S(f) = \text{Fourier}\{R(\tau)\} = \frac{X(f) X^*(f)}{T} \).

1. Use a pseudo-random number generator to generate a single sequence of 6000 numbers with a Gaussian normal distribution \( N(\mu_x=0, \sigma_x^2=1) \) that has zero mean and variance of 1. Consider these data to represent random sea level oscillations at Lewes, DE in meters sampled with \( \Delta t=1 \) hour. Compute and graph estimates of

(a) the auto-spectral power density \( S_{\text{noise}}(f) \) with 95% confidence limits for 20 degrees of freedom.

2. Suppose that the Lewes sealevel oscillations also contain tidal signals in addition to the noise, that is,

\[
x(t) = N(\mu_x=0, \sigma_x^2=1) + a_1 \cos(w_1 t) + a_2 \cos(w_2 t) + a_3 \cos(w_3 t)
\]

where \( w_i = 2\pi/T_i \) with \( T_i = \{72.00, 12.00, 12.42\} \) hours and \( a_i = \{0., 0.2, 0.5\} \) meters for \( i = \{1, 2, 3\} \). You are asked to sample this time series at \( \Delta t=1 \) hour for a total record length of 6000 hours. As in (1), compute and graph estimates of

(b) the auto-spectral power density \( S_{\text{raw}}(f) \) with 95% confidence limits for 20 degrees of freedom.

3. Write a program to implement a box car filter and use this filter to extract the low-frequency or 72-hour signal. As in (1) and (2) above, compute and graph estimates of

(c) the auto-spectral power density \( S_{\text{filter}}(f) \) with 95% confidence limits for 20 degrees of freedom;

(d) Do you expect noise to contaminate your filtered output?

4. Estimate the coherence squared \( \Gamma^2(f) \) and its 95% confidence level, transfer function, and phase between the raw and filtered data for 20 degrees of freedom. Note that these properties relate to cross-spectral estimates.

Please describe what you have done, how you have done it, and what you have learnt from these exercises. Please attach your programs to your write-up, and properly label all axes and their units are correct. Please make sure to mention properties related to sampling rate and record length in both time and frequency domains.
Often students fail to recognized that the Fourier Transform of the lagged auto-correlation $R_x(\tau)$ is the power-spectra of the simulated sealevel oscillations $x(t)$, that is

$$S(f) = \text{Fourier}\{R(\tau)\} = X(f) \ast X^*(f)/T$$

Where $X(f) = \text{Fourier}\{x(t)\}$ and $T$ is the record length. Hence I here note that it may help your interpretation, if you compare the raw and filtered time series and spectra of $x(t)$, perhaps even taking the ratio of filtered and raw spectra ($S_{\text{filter}}/S_{\text{raw}}$).

Furthermore, often students give insufficient interpretations of the values shown for the spectra. I suspect that this is caused by the presence of Gaussian noise that may hide the signals. For the purpose of this optional addendum to your work, you may consider the time series without the Gaussian noise

$$x(t) = a_1 \cos(w_1 t) + a_2 \cos(w_2 t) + a_3 \cos(w_3 t)$$

with the same coefficients $a_i$ and $w_i$ as in the original assignment.

As always, please scrutinize, discuss, and interpret units and values of plots in the write-up of your analyses. I would like you to think and understand what you have done rather than just handing in plots. What you think is much more important to me than what you show. A critical evaluation of your own work can include a suspicion where something may be wrong.