1999 Geophysical Data Analysis
Solution to homework #1

The periodic function

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ \sin t & 0 < t < \pi \end{cases}$$

with period $T = 2\pi$ shall be expanded into a Fourier series

$$\sum_{n=-\infty}^{\infty} C_n e^{int} \qquad C_n \text{ complex}$$

The coefficients C_n are found from

$$C_n = \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \qquad n = -\infty, \dots 0 \dots +\infty$$

$$= \int_0^{\pi} \sin t\, e^{-int} = \int_0^{\pi} \frac{e^{it} - e^{-it}}{2i} \cdot e^{-int}\, dt$$

$$= -\frac{i}{2} \int_0^{\pi} e^{i(n+1)t} - e^{i(n-1)t}\, dt$$

there is a sign error in here that probably "cancels" on account of squaring.
AM
9/29/03

$$= -\frac{i}{2} \left\{ \frac{1}{i(n+1)} \cdot e^{i(n+1)t} - \frac{1}{i(n-1)} e^{i(n-1)t} \right\} \Bigg|_0^{\pi} \qquad |n| \gtrless 1$$

For $n = 1$:

$$C_1 = -\frac{i}{2}\int_0^{\pi} e^{2it} - 1 \quad dt = -\frac{i}{2}\left\{ \frac{1}{2i} e^{2it} - t \right\}\Big|_0^{\pi}$$

$$= -\left(\frac{1}{4} e^{2it} - \frac{i}{2} t \right)\Big|_0^{\pi} = -\left(\frac{1}{4} e^{2\pi i} - \frac{1}{4} - \frac{\pi}{2} i \right)$$

$$C_1 = -\left(\frac{1}{4} \cos 2\pi - \frac{1}{4} - \frac{\pi}{2} i \right) = \overset{+}{-} \frac{\pi}{2} i$$

(1) $\boxed{C_1 = \overset{+}{-} \frac{\pi}{2} i}$

For $n = -1$:

$$C_{-1} = -\frac{i}{2}\int_0^{\pi} 1 - e^{-2it} \quad dt = -\frac{i}{2}\left\{ t + \frac{1}{2i} e^{-2it} \right\}\Big|_0^{\pi}$$

$$= -\frac{i}{2}\left\{ \pi + \frac{1}{2i} e^{-2\pi i} - \frac{1}{2i} \right\} = -\frac{i\pi}{2} - \frac{1}{4}\cos 2\pi + \frac{1}{4}$$

(2) $\boxed{C_{-1} = -\frac{i\pi}{2}}$

For $n = 0$: $\quad C_0 = -\frac{i}{2}\int_0^{\pi} e^{it} - e^{-it} \, dt = \int_0^{\pi} \sin t \, dt = -\cos t\Big|_0^{\pi} =$

$$= -(-1-1) = 2$$

for $|n| > 1$

$$C_n = -\frac{1}{2}\left\{ \frac{e^{i(n+1)\pi} - e^{i(n+1)\cdot 0}}{(n+1)} - \frac{e^{i(n-1)\pi} - e^{i(n-1)0}}{n-1} \right\}$$

$$= -\frac{1}{2}\left\{ \frac{\cos[(n+1)\pi] - 1}{(n+1)} - \frac{\cos[(n-1)\pi] - 1}{(n-1)} \right\}$$

for n integer $\cos(n\pi + \pi) = \cos(n\pi - \pi) = (-1)^{|n|+1} = (-1)^{|n|+1}$

then

$$C_n = -\frac{1}{2}\left\{ \frac{(-1)^{|n|-1}}{n+1} - \frac{(-1)^{|n|+1}}{n-1} \right\}$$

$$C_n = -\frac{1}{2} \cdot \left\{ (-1)^{|n|+1} \frac{(n-1) - (n+1)}{n^2-1} = -\frac{1}{2}(-1)^{|n|+1} \cdot \frac{-2}{n^2-1} = \right.$$

(3) $$\boxed{C_n = \frac{(-1)^{|n|+1}}{n^2-1}}$$

It is therefore seen that $\boxed{C_n = C_{-n}}$

$$\cancel{f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{int} = 1 + \sum_{n=1}^{\infty} C_n\left(e^{int} + e^{-int}\right) +}$$

Then the Fourier expansion of the function $f(t)$ is

$$f(t) = 2 + \frac{\pi}{2} i \left(e^{it} - e^{-it} \right) + \sum_{n=2}^{\infty} c_n \left(e^{int} + e^{-int} \right)$$

$$= 2 - \pi \frac{e^{it} - e^{-it}}{2i} + \sum_{n=2}^{\infty} c_n \cdot 2 \frac{e^{int} + e^{-int}}{2}$$

$$\boxed{\Rightarrow f(t) = 2 - \pi \sin t + 2 \cdot \sum_{n=2}^{\infty} c_n \cos nt \qquad \text{with } c_n = \frac{(-1)^{n-1}}{n^2 - 1} \quad n > 1}$$

(b)

$$f\left(t = \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 2 - \pi \sin \frac{\pi}{2} + 2 \cdot \sum_{n=2}^{\infty} c_n \cos \frac{n\pi}{2}$$

$$\Rightarrow \quad 1 = 2 - \pi + 2 \sum_{n=2}^{\infty} c_n \cos \frac{n\pi}{2}$$

$$\Rightarrow \quad \pi - 2 = -1 + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2 - 1} \cos \frac{n\pi}{2}$$

$m = \frac{n}{2}$

$$\Rightarrow \quad \frac{\pi - 2}{4} = -\frac{1}{4} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{2m-1}}{4m^2 - 1} \cos m\pi$$

because $\cos \frac{n\pi}{2} = 0$ for n odd $\Rightarrow$ only contributions for n even: $2, 4, 6 \ldots$ $\Rightarrow$ $m = 1, 2, 3 \ldots$

then

$$\frac{\pi-2}{4} = -\frac{1}{4} + \frac{1}{2}\sum_{m=1}^{\infty} \frac{(-1)}{4m^2-1} \cdot (-1)^m$$

$$= -\frac{1}{4} + \frac{1}{2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{4m^2-1}$$

so $\pi - 2 = -\frac{1}{\cancel{4}} + \cancel{\frac{1}{2}}2\left\{ \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} \dots \right\}$

so $\pi = 1 + 2 \left\{ - '' - \right\}$

or

$$\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \frac{1}{99} \dots = \frac{\pi-2+1}{4} \cdot 2 = \frac{2\pi-2}{4} = \frac{\pi-1}{2}$$

$$\underset{m=1}{\frac{1}{3}} - \underset{m=2}{\frac{1}{3\cdot 5}} + \underset{m=3}{\frac{1}{5\cdot 7}} - \underset{m=4}{\frac{1}{7\cdot 9}} + \underset{m=5}{\frac{1}{9\cdot 11}}$$

$4m^2-1$

$$\pi = 2\left(\dots \right) + 1$$

$$\pi = 1 + 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \frac{1}{99} \dots \right)$$

$2\cdot 2-1$ $\quad\cdot 8-1$ $\quad 0\cdot 10-1$

(2) (a) The Fourier transform $F(\omega)$ of the function

$$f(t) = \begin{cases} 0 & t < 0 \\ \beta e^{-\alpha t} & t > 0 \end{cases}$$

shall be found.
The F.T. is defined as

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

Inserting the given fct'n

$$F(\omega) = \int_0^\infty \beta e^{-\alpha t} e^{-i\omega t}\, dt = \beta \int_0^\infty e^{-(\alpha+i\omega)t}\, dt$$

$$= \beta \lim_{\varepsilon\to\infty} \int_0^\varepsilon e^{-(\alpha+i\omega)t}\, dt = \beta \lim_{\varepsilon\to\infty} \left\{ -\frac{1}{(\alpha+i\omega)}\, e^{-(\alpha+i\omega)t} \Big|_0^\varepsilon \right.$$

$$= \beta \lim_{\varepsilon\to\infty} (-) \frac{1}{(\alpha+i\omega)} \left(e^{-(\alpha+i\omega)\varepsilon} - 1 \right)$$

✓

$$= \frac{\beta}{\alpha+i\omega} \lim_{\varepsilon\to\infty} \left(1 - e^{-(\alpha+i\omega)\varepsilon}\right) = \frac{\beta}{\alpha+i\omega}$$

The inverse Fourier transform of $F(\omega)$ is

$$f(t) = \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega = \beta \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\alpha + i\omega}\, d\omega$$

$$= \beta \int_{-\infty}^{+\infty} \frac{(\alpha - i\omega)\, e^{i\omega t}}{\alpha^2 + \omega^2}\, d\omega = \beta \int_{-\infty}^{+\infty} \frac{(\alpha - i\omega)\,(\cos\omega t + i \sin\omega t)}{\alpha^2 + \omega^2}\, dt$$

$$f(t)/\beta = \int_{-\infty}^{+\infty} \frac{\alpha \cos\omega t + \omega \sin\omega t}{\alpha^2 + \omega^2}\, d\omega + i \int_{-\infty}^{+\infty} \frac{-\omega \cos\omega t + \alpha \sin\omega t}{\alpha^2 + \omega^2}\, dt$$

$$= \underbrace{\int_{-\infty}^{+\infty} \frac{\alpha \cos\omega t}{\alpha^2 + \omega^2}\, d\omega} + \underbrace{\int_{-\infty}^{+\infty} \frac{\omega \sin\omega t}{\alpha^2 + \omega^2}\, d\omega} + i \underbrace{\int_{-\infty}^{+\infty} \frac{-\omega \cos\omega t}{\alpha^2 + \omega^2}\, d\omega} + i \underbrace{\int_{-\infty}^{+\infty} \frac{\alpha \sin\omega t}{\alpha^2 + \omega^2}\, d\omega}$$

$$= \pi e^{-|\alpha t|} + \pi e^{-|\alpha t|} \cdot \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} + i \cdot 0 + i \cdot 0$$

$$= 2\pi e^{-|\alpha t|} \cdot \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \qquad \text{q.e.d.}$$

The integrals are evaluated in the appendix with the aid of the following "known" integrals (Bronstein + Semendjajew, 1981):

$$\int_0^\infty \frac{\cos ax}{1 + x^2}\, dx = \frac{\pi}{2}\, e^{-|a|}$$ # 18 on page 120 there

$$\int_0^\infty \frac{x \sin bx}{a^2 + x^2}\, dx = \frac{\pi}{2}\, e^{-|a \cdot b|} \cdot \begin{cases} 1 & \text{for } b > 0 \\ -1 & \text{for } b < 0 \end{cases}$$ # 17 on page 120 there

I also reference a hint from B. McCarthy, that it is possible to solve the integral without Residues

$$(1)\quad \int_{-\infty}^{+\infty} \frac{\alpha\cos\omega t}{\alpha^2+\omega^2}\,d\omega = 2\int_{0}^{\infty} \frac{\alpha\cos\omega t}{\alpha^2+\omega^2}\,d\omega = 2\int_0^\infty \frac{\cancel{\alpha}}{\alpha^{\cancel{2}}}\cdot\frac{\cos\omega t}{1+\frac{\omega^2}{\alpha^2}}\,d\omega = 2\int_0^\infty \frac{\cos\alpha t\omega}{1+s^2}\,ds$$

$s = \frac{\omega}{\alpha}$

$ds = \frac{1}{\alpha}\,d\omega$

$$= \pi e^{-|\alpha t|}$$

Bronstein #19 ✓

$$(4)\quad \int_{-\infty}^{+\infty} \frac{\alpha\sin\omega t}{\alpha^2+\omega^2}\,d\omega = \int_{-\infty}^{0} \frac{\alpha\sin\omega t}{\alpha^2+\omega^2}\,d\omega + \int_0^\infty \frac{\alpha\sin\omega t}{\alpha^2+\omega^2}\,d\omega = \int_\infty^0 \alpha\frac{\sin(-\omega t)}{\alpha^2+\omega^2}(-)\,ds = \int_0^\infty \frac{\alpha\sin(-st)}{\alpha^2+s^2}\,ds + \int_0^\infty \frac{\alpha\sin\omega t}{\alpha^2+\omega^2}\,d\omega$$

$s = -\omega$

$ds = -d\omega$

$$= \int_0^\infty \frac{-\alpha\sin\omega t}{\alpha^2+\omega^2}\,d\omega + \int_0^\infty \frac{\alpha\sin\omega t}{\alpha^2+\omega^2} = 0$$

$\sin -st = -\sin st$ ✓

$$(3)\quad \int_{-\infty}^{+\infty} \frac{-\omega\cos\omega t}{\alpha^2+\omega^2}\,d\omega = \int_0^{\infty} \frac{-\omega\cos\omega t}{\alpha^2+\omega^2}\,d\omega + \int_{-\infty}^{0} \frac{-\omega\cos\omega t}{\alpha^2+\omega^2} = \int_0^{\infty} \frac{-\omega\cos\omega t}{\alpha^2+\omega^2}\,d\omega + \int_{\infty}^{0} \frac{s\cos st}{\alpha^2+s^2}(-)\,ds$$

$s = -\omega$

$ds = -d\omega$

$$= \int_0^{\infty} \frac{-\omega\cos\omega t}{\alpha^2+\omega^2}\,d\omega + \int_0^{\infty} \frac{\omega\cos\omega t}{\alpha^2+\omega^2} = 0$$

$$(2)\quad \int_{-\infty}^{+\infty} \frac{\omega\sin\omega t}{\alpha^2+\omega^2}\,d\omega = \int_0^{\infty} \frac{\omega\sin\omega t}{\alpha^2+\omega^2}\,d\omega + \int_{-\infty}^{0} \frac{\omega\sin\omega t}{\alpha^2+\omega^2}\,d\omega = \int_0^{\infty} \frac{\omega\sin\omega t}{\alpha^2+\omega^2} + \cancel{(-)} \int_{\cancel{\infty}\,0}^{\cancel{0}\,\infty} \frac{-s\sin(st)}{\alpha^2+\omega^2}\,d\omega$$

$s = -\omega$

$ds = -d\omega$

$$= \int_0^{\infty} \frac{\omega\sin\omega t}{\alpha^2+\omega^2}\,d\omega + \int_0^{\infty} \frac{\omega\sin\omega t}{\alpha^2+\omega^2}\,d\omega = 2\int_0^{\infty} \frac{\omega\sin\omega t}{\alpha^2+\omega^2}\,d\omega = \pi\cdot e^{-|\alpha t|}\cdot\begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$$

Bronstein #17