Bendat + Piersol (1986)
Chapt. 3+4

We want to estimate variance in time and frequency domains and estimate confidence limits on these estimates

→ show example of 9-year tide record Discovery Harbor

PROBABILITY FUNDAMENTALS

Def.: random variable $\{x\} = \{x_1, x_2, \ldots x_n\}$ set

Def. Probability distribution $P(x) = \text{Prob}\left[x_k \leq x\right]$

Def. Probability density function (histogram $p(x) = \lim_{\Delta x \to 0} \frac{\text{Prob}\left[x < x_k \leq x + \Delta x\right]}{\Delta x}$

$$p(x) \geq 0 \quad ; \quad \int_{-\infty}^{+\infty} p(x)\,dx = 1 \quad ; \quad P(x) = \int_{-\infty}^{x} p(x')\,dx'$$

A random variable is completely specified by its pdf $p(x)$

p. 54 sketch here

Def. Expected value $E[x] \equiv \langle x \rangle = \int_{-\infty}^{+\infty} x \cdot p(x)\,dx = \mu_x$ 1st moment
(mean, average, most likely)

similar $E[g(x)] = \int_{-\infty}^{+\infty} g(x) \cdot p(x)\,dx$

For $g(x) = x^2$, the mean or averaged, or expected square value (variance)

$$E[x^2] = \int_{-\infty}^{+\infty} x^2 \, p(x) \, dx = \sigma_x^2 \qquad 2^{nd} \text{ statistical moment}$$

Then the variance of the set $\{x\}$ is

$$E[(x-\mu_x)^2] = \int_{-\infty}^{+\infty} (x-\mu_x)^2 \, p(x) \, dx$$

$$= \int x^2 p(x) \, dx + \int (-) \, 2 x \mu_x \, p(x) \, dx + \int \mu_x^2 \, p(x) \, dx$$

$$= \sigma_x^2 - 2\mu_x \cdot \mu_x + \mu_x^2$$

$$= \sigma_x^2 - \mu_x^2$$

Two random variables $\{x\}$ and $\{y\}$

Def. $P(x,y) \equiv \text{Prob}[x_k \leq x \text{ and } y_k \leq y]$

Def. $p(x,y) \equiv \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\text{Prob}[x < x_k < x+\Delta x \text{ and } y < y_k < y + \Delta y]}{\Delta x \, \Delta y}$

Def. $p(x,y) \geq 0 \; ; \quad \iint_{-\infty}^{+\infty} p(x,y) \, dx \, dy = 1$

Def. $P(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p(x', y') \, dy' \, dx'$

If $p(x,y) = p(x) \cdot p(y)$

Then x and y are independent (uncorrelated) random variables

The expectant operator is linear, that is

$$E[x+y] = E[x] + E[y]$$

provided that $\{x\}$ and $\{y\}$ have the same joint pdf $p(x,y)$

$\boxed{\text{Central Limit Theorem}}$

Define random variable $G = x_1 + x_2 + x_3 + \ldots + x_N$

where x_i are mutually independent random variables

with $E[x_i x_j] = 0 \quad i \neq j$ and unspecified and possibly different pdf

The mean of this variable

$$\mu_G = E[G] = E\left[\sum_{i=1}^{N} x_i\right] = \sum_{i=1}^{N} E[x_i] = \sum_{i=1}^{N} \mu_i$$

$$\sigma_G^2 = E[(G-\mu_G)^2] = E\left[\left(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \mu_i\right)^2\right] = E\left[\left(\sum_{i=1}^{N} x_i - \mu_i\right)^2\right]$$

$$= \sum_{i=1}^{N} E[(x_i-\mu_i)^2] \quad \text{because } E[x_i x_j] = 0 \quad i \neq j$$

$$= \sum_{i=1}^{N} \sigma_i^2$$

As $N \to \infty$, the variable G with μ_G and σ_G becomes Gaussian. [No proof]