p.46 $\quad \text{Prob}[x_k \leq x] = \int_{-\infty}^{x} p(x)\,dx' = \int_{-\infty}^{\chi_n^2} p(\chi^2)\,d\chi^2$

p.49 $\quad$ where $p(\chi_n^2) = \dfrac{(\chi_n^2)^{n/2-1}\, e^{-\chi_n^2/2}}{2^{n/2}\,\Gamma(n/2)}$

$\Gamma(n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)$ gamma fctn.

p.54 Hypothesis thesting:

p-54 of notes (Bendat + Piersol p.88-91)

p.70 Separate the original time series of $m \cdot T$ length into "m" pieces each with a record length $T = N \cdot \Delta t$

Then the average of these "m" periodigrams is

$$\overline{\hat{P}_n} = \frac{1}{m} \sum_{k=1}^{m} \cancel{|X_{nk}|^2}\ \hat{P}_{n,k}$$

$P_n = \frac{1}{T} X_n \cdot X_n^*$

$X_n = \Delta t \sum_k x_k e^{-j2\pi \frac{\ell}{n}}$

For $m = 2$:

$$\overline{\hat{P}_n} = \underbrace{\frac{\Delta t^{\cancel{2}}}{N \cancel{\Delta t}}}_{\Delta t / 2N} \cdot \frac{1}{2} \left(A_{n,1}^2 + A_{n,2}^2 + B_{n,1}^2 + B_{n,2}^2 \right)$$

p.71 $\quad \ldots \quad \dfrac{\overline{\hat{P}_n}}{\Delta t/2} \cdot \dfrac{1}{\sigma_x^2/2} = \dfrac{A_{n,1}^2 + A_{n,2}^2 + B_{n,1}^2 + B_{n,2}^2}{N \sigma_x^2/2}$

$= \chi_4^2$ as each term is $N(0,1)$

$\underbrace{2m\,\overline{\hat{P}_n}}\,/\,\underbrace{P_n}$

$$E[\,4\,\overline{\hat{P}_n}/(\Delta t\,\sigma_x^2)\,] = E[\chi_4^2] = 4 \qquad = \frac{4}{\cancel{\Delta t}\,\sigma_x^2}\,E[\overline{\hat{P}_n/\Delta t}]$$

$$E[\overline{\hat{P}_n/\Delta t}] = \sigma_x^2 = P_n/\Delta t$$

So generally for m segments

$$\frac{2m\,\overline{\hat{P}_n}}{P_n} = \chi^2_{2m} \qquad \text{where } \frac{P_n}{\Delta t} = \sigma_x^2$$

2m degrees of freedom

$$\mathrm{Prob}[\chi^2_{2m} > \chi^2_{2m,\alpha}] = \alpha = \int_{-\infty}^{\chi^2_{2m,\alpha}} p(\chi^2)\,d\chi^2$$

$$\mathrm{Prob}[\chi^2_{2m} > \chi^2_{2m,1-\alpha}] = 1-\alpha = \int_{-\infty}^{\chi^2_{2m,1-\alpha}} p(\chi^2)\,d\chi^2$$

$\downarrow$

$$\mathrm{Prob}[\chi^2_{2m,\alpha} \geq \chi^2_{2m} > \chi^2_{2m,1-\alpha}] = \mathrm{Prob}[\chi^2_{2m} > \chi^2_{2m,1-\alpha}]$$

$$- \mathrm{Prob}[\chi^2_{2m} > \chi^2_{2m,\alpha}] =$$

$$= 1 - 2\alpha$$

Or with $\chi^2_{2m} = 2m\,\overline{\hat{P}_n}/P_n$

[redacted]

$$1 - 2\alpha = \text{Prob}\left[\chi^2_{2m,\alpha} \geq 2m \frac{\overline{\hat{P}_n}}{P_n} > \chi^2_{2m,1-\alpha} \right]$$

$$= \text{Prob}\left[\frac{1}{\chi^2_{2m,\alpha}} \leq \frac{P_n}{\overline{\hat{P}_n} \cdot 2m} < \frac{1}{\chi^2_{2m,1-\alpha}} \right]$$

$$= \text{Prob}\left[\frac{2m\,\overline{\hat{P}_n}}{\chi^2_{2m,\alpha}} \leq P_n < \frac{2m\,\overline{\hat{P}_n}}{\chi^2_{2m,1-\alpha}} \right]$$

For $\alpha = 0.025$:
$m = 10$

$$0.95 = \text{Prob}\left[0.59\,\overline{\hat{P}_n} \leq P_n \leq 2.09\,\overline{\hat{P}_n} \right]$$

$$= \text{Prob}\left[\log(0.59\,\overline{\hat{P}_n}) \leq \log(P_n) < \log(2.09\,\overline{\hat{P}_n}) \right]$$

$$= \text{Prob}\left[\underbrace{\log 0.59}_{\text{lower bound}} + \underbrace{\log \overline{\hat{P}_n}}_{\text{estimate}} \leq \underbrace{\log P_n}_{\text{true value}} < \underbrace{\log 2.09}_{\text{upper bound}} + \underbrace{\log \overline{\hat{P}_n}}_{\text{estimate}} \right]$$

95% confidence

$$= \text{Prob}\left[\log 0.59 \leq \log P_n - \log \overline{\hat{P}_n} < \log 2.09 \right]$$

$$= \text{Prob}\left[-0.23 \leq \log\left(P_n / \overline{\hat{P}_n} \right) < 0.32 \right]$$

$$E[\chi_n^2] = n$$ Chi-Square Distribution

$$E\left[2\frac{\hat{P}_n}{\Delta t\,\sigma_x^2}\right] = 2 = E[\chi_2^2]$$ Normalized Periodogram

$$E\left[\frac{\hat{P}_n}{\Delta t}\right] = \sigma_x^2 \equiv \frac{P_n}{\Delta t}$$ <u>Definition</u> of P_n as an unbiased variable

$$E[\hat{P}_n] = P_n = \Delta t \cdot \sigma_x^2$$

$$\overline{\hat{P}}_n = \frac{1}{m}\sum_{l=1}^{m} \hat{P}_{n,l}$$

$$\frac{f+g}{H} = \text{const}$$

$$\frac{g}{H} = \text{const}$$

$$\left.\frac{\partial v}{\partial x}\right|_{H(y)} \frac{\partial g}{\partial y} = \text{const}$$

$$E\left[2m\frac{\overline{\hat{P}}_n}{\cancel{\Delta t\,\sigma_x^2}}\right] = 2m = E[\chi_{2m}^2]$$

$$\overline{\hat{P}}_n / P_n$$

$$\boxed{2m\frac{\overline{\hat{P}}_n}{P_n} = \chi_{2m}^2}$$

$$\text{Prob}\left[\frac{2m\,\overline{\hat{P}}_n}{\chi^2_{2m,\alpha}} \le P_n < \frac{2m\,\overline{\hat{P}}_n}{\chi^2_{2m,\alpha}}\right] = 1-2\alpha$$

For $\alpha = 0.025$ (95% confidence) $m = 10$

$$0.95 = \text{Prob}\left[0.59\,\overline{\hat{P}}_n \le P_n < 2.09\,\overline{\hat{P}}_n\right]$$ take log

$\log(0.59) = -0.23$ $\log 2.09 = 0.32$ $\log(a\cdot b) = \log a + \log b$

$$= \text{Prob}\left[-0.23 \le \log P_n/\overline{\hat{P}}_n < 0.32\right]$$