Chapter 3 The Fourier Transform

3.1 Introduction

There are two main approaches to Fourier transform theory

1. Define the Fourier transform of a function $f(t)$ as
$$F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-\mathrm{j}2\pi\nu t)\, \mathrm{d}t \tag{3.1}$$
and show that under suitable conditions $f(t)$ can be recovered from $F(\nu)$ via the inverse transform relationship
$$f(t) = \int_{-\infty}^{\infty} F(\nu) \exp(+\mathrm{j}2\pi\nu t)\, \mathrm{d}\nu \tag{3.2}$$
This can be motivated in terms of finding the eigenvalues of a linear time-invariant system, as discussed in the previous chapter. The calculation of $F(\nu)$ from $f(t)$ is called **Fourier analysis**, while the recovery of $f(t)$ from $F(\nu)$ is called **Fourier synthesis**.
2. Start from the Fourier series which expresses a periodic function of t as a sum of cosinusoidal and sinusoidal functions and let the period T become large.

The first approach is more satisfactory mathematically but the second is somewhat more easily motivated physically. We shall adopt the first approach and derive the second via generalized function theory.

3.2 The Fourier transform and its inverse

Historically, people were interested whether or not $f(t)$ could be successfully recovered from $F(\nu)$ on a point-by-point basis. Thus if we define the Fourier transform by (3.1) and then calculate
$$f_M(t) = \int_{-M}^{M} F(\nu) \exp(\mathrm{j}2\pi\nu t)\, \mathrm{d}\nu \tag{3.3}$$
we would like to show that $f_M(t) \to f(t)$ as $M \to \infty$.

Substituting (3.1) into the equation for $f_M(t)$ yields
$$f_M(t) = \int_{-M}^{M} \left[\int_{-\infty}^{\infty} f(\tau) \exp(-\mathrm{j}2\pi\nu\tau)\, \mathrm{d}\tau\right] \exp(\mathrm{j}2\pi\nu t)\, \mathrm{d}\nu \tag{3.4}$$
$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-M}^{M} \exp\left[\mathrm{j}2\pi\nu\,(t-\tau)\right]\, \mathrm{d}\tau\right] \mathrm{d}\tau \tag{3.5}$$
$$= \int_{-\infty}^{\infty} f(\tau) \frac{\sin\left[2\pi M\,(t-\tau)\right]}{\pi\,(t-\tau)}\, \mathrm{d}\tau \tag{3.6}$$

This is called a **Dirichlet integral**. It is a weighted average of $f(\tau)$ with a $\sin(Mx)/x$ weighting centred about $\tau = t$. Equivalently, we may regard it as the **convolution** of f with the function

$$h_M(t) = \frac{\sin(2\pi M t)}{\pi t}$$

If we can show that as M tends to infinity $h_M(t) \to \delta(t)$, then $(f * h_M) \to (f * \delta) = f$, so the inverse transform will recover $f(t)$ (see Figure 3.1).

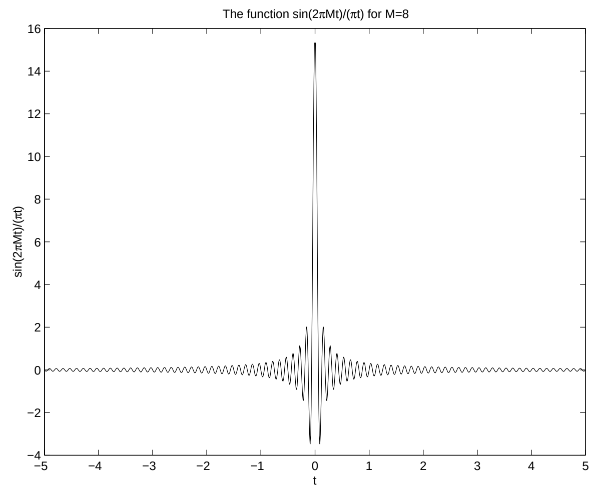


Figure 3.1 Function $h_M(t)$ with which $f(t)$ is convolved when evaluating the inverse transform integral using integration limit $M = 8$

(*Technical note:* It is easy to show that the Dirichlet integral evaluates to $f(t)$ provided f is well behaved, e.g. if f is differentiable at t (as we shall show later). However, we usually want f to be less well behaved than this and the result becomes harder to prove. In fact, even as strong a condition as the continuity of f at t is **not** sufficient to make the inverse Fourier transform converge at t.)

Notes:

1. The Fourier transform transforms one **complex** function of a **real** variable into another **complex** function of a (different) **real** variable.

2. Other definitions of the Fourier transform are also in use which differ in scaling. A common alternative, particularly when associated with the Laplace transform, is

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \tag{3.7}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \tag{3.8}$$

and yet another version is similar but with a $1/\sqrt{2\pi}$ factor in front of both integrals. Our version keeps most of the 2π's in the exponential factor where they belong and simplifies and rationalises the scaling in many other places. However one must be able to cope with the other forms because is no general agreement as to which definition should be used. Another (less common) variation is to associate the negative sign in the exponent with the inverse rather than the forward transform.

3.2.1 Existence and invertibility of the Fourier transform

Many conditions can be used to describe classes of functions which have classical Fourier transforms (i.e., the Fourier transform should exist everywhere and be finite). It is necessary to further restrict this class if we want to be able to recover the original function by using the inverse transform formula.

One sufficient (though not necessary) condition (due to Jordan) is that if f is in $\mathcal{L}^1$ (i.e., f is **absolutely integrable**) and if it is of **bounded variation** on every finite interval, then $F(\nu)$ exists and $f(t)$ can be recovered from the inverse Fourier transform relationship at each point at which f is **continuous**.

The first condition (that f is in $\mathcal{L}^1$) means that

$$\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty \tag{3.9}$$

while the second (bounded variation) means that $f(t)$ can be expressed as the difference of two bounded, monotonic increasing functions and excludes functions like $t\sin(1/t)$.

If $f(t)$ is discontinuous at $t = t_0$, the inverse Fourier transform integral still converges at t_0 and its value there is

$$\frac{1}{2}\left[f\left(t_0^+\right) + f\left(t_0^-\right)\right] \tag{3.10}$$

(*Technical note:* The right and left-hand limits are guaranteed to exist for functions of bounded variation.)

Notes:

1. The **existence** of the Fourier transform is guaranteed if f is just absolutely integrable. Bounded variation in a neighbourhood of a point is needed for the inverse transform to recover the value of the function at that point (or the middle of the jump as in (3.10) if the function is discontinuous there).

2. The above approach to Fourier transforms is asymmetrical in the sense that the Fourier transforms of many absolutely integrable functions are not absolutely integrable and so do not lie in the original space of functions. For example, as we shall see later, the Fourier transform of the absolutely integrable "top-hat" function $\Pi(t)$ defined by

$$\Pi(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

   is $\mathrm{sinc}(\nu)$. Although $\mathrm{sinc}(\nu)$ is bounded, it is not absolutely integrable. The inverse transform integral in this case has to be interpreted somewhat differently from that in the Fourier transform. Technically, when the integral in the Fourier transform is taken as a Lebesgue integral, that in the inverse Fourier transform is an improper Riemann integral which may only exist in the sense of the Cauchy principal value.

   Of course, **if** the Fourier transform of the function **does** happen to be absolutely integrable, the inverse transform integral can be taken as a standard Lebesgue integral as well.

3. An alternative approach is to restrict the class of functions to be the square integrable functions (i.e. the so called $\mathcal{L}^2$ functions) for which

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t < \infty \tag{3.12}$$

It can be shown that the Fourier transform of an $\mathcal{L}^2$ function is guaranteed to be another $\mathcal{L}^2$ function. This leads to a more symmetrical theory but in this case the transform and the inverse transform do not necessarily converge pointwise but may do so only in the $\mathcal{L}^2$ sense.

4. Both of the above approaches are too restrictive for many practical applications because we want to be able to take Fourier transforms of a wider class of functions and to be sure that the spaces of the original and transformed functions coincide. As we shall see later, considering generalized functions allows us to do just this and to simplify greatly the conditions for existence of the Fourier transform.

3.3 Basic properties

If $F(\nu)$ is the Fourier transform of $f(t)$ we will write $f(t) \leftrightarrow F(\nu)$ etc. It can generally be assumed that a function denoted by a capital (upper case) letter is the Fourier transform of the function denoted by the corresponding small (lower case) letter.

Proofs of the following properties, where omitted, should be done as exercises. A common step is to interchange the order of integration and we will assume that this is allowed, but with the understanding that in each case the functions must be sufficiently well behaved. These properties also apply for Fourier transforms of generalized functions (to be defined later) except where explicitly stated.

3.3.1 Linearity

$$c_1 f_1(t) + c_2 f_2(t) \leftrightarrow c_1 F_1(\nu) + c_2 F_2(\nu) \tag{3.13}$$

for any real or complex constants c_1 and c_2.

3.3.2 Behaviour under complex conjugation

$$f^*(t) \leftrightarrow F^*(-\nu) \tag{3.14}$$

Note the reversal of the frequency axis.

3.3.3 Duality

$$F(t) \leftrightarrow f(-\nu) \tag{3.15}$$

For every forward Fourier transform there is a corresponding dual inverse transform which is almost the same except for a sign reversal.

Note that if we are not working with $\mathcal{L}^2$ functions or generalized functions, we must add the condition that the appropriate transforms exist.

3.3.4 Shifting

$$f(t - t_0) \leftrightarrow \exp(-\mathrm{j}2\pi\nu t_0)F(\nu) \tag{3.16}$$
$$\exp(+\mathrm{j}2\pi\nu_0 t)f(t) \leftrightarrow F(\nu - \nu_0) \tag{3.17}$$

Shifting a function in one domain has no effect on the **magnitude** of the corresponding function in the other domain but affects only its **phase**. The amount of phase shift varies linearly and the slope depends on the amount of shift in the other domain.

3.3.5 Modulation

$$f(t)\cos(2\pi\nu_0 t) \leftrightarrow \frac{1}{2}\left[F\left(\nu+\nu_0\right) + F\left(\nu-\nu_0\right)\right] \tag{3.18}$$
$$f(t)\sin(2\pi\nu_0 t) \leftrightarrow \frac{\mathrm{j}}{2}[F(\nu+\nu_0) - F(\nu-\nu_0)] \tag{3.19}$$

3.3.6 Interference

$$f(t-t_0) + f(t+t_0) \leftrightarrow 2\cos(2\pi\nu t_0)F(\nu) \tag{3.20}$$

3.3.7 Scaling

$$f(at) \leftrightarrow \frac{1}{|a|}\,F\left(\frac{\nu}{a}\right) \tag{3.21}$$
$$\frac{1}{|b|}\,f\left(\frac{t}{b}\right) \leftrightarrow F(b\nu) \tag{3.22}$$

The form is the same in each direction. Factors a and b must be **real**. Note in particular that

$$f(-t) \leftrightarrow F(-\nu) \tag{3.23}$$

3.3.8 Differentiation

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leftrightarrow \mathrm{j}2\pi\nu\, F(\nu) \tag{3.24}$$
$$-\mathrm{j}2\pi t\, f(t) \leftrightarrow \frac{\mathrm{d}F(\nu)}{\mathrm{d}\nu} \tag{3.25}$$

Exercise: Show that if $t^k f(t)$ is absolutely integrable for all $0 \le k \le n$ then the Fourier transform $F(\nu)$ is differentiable n times.

3.3.9 Integration

Provided that

$$\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = 0 \tag{3.26}$$

we find that

$$\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau \leftrightarrow \frac{1}{\mathrm{j}2\pi\nu} F(\nu) \tag{3.27}$$

Similarly **provided that**

$$\int_{-\infty}^{\infty} F(\nu)\,\mathrm{d}\nu = 0, \tag{3.28}$$

$$-\frac{1}{\mathrm{j}2\pi t} f(t) \leftrightarrow \int_{-\infty}^{\nu} F(\mu)\,\mathrm{d}\mu \tag{3.29}$$

If the integrals over the entire range of the variables do **not** vanish, we must use generalized functions in the transforms, as will be discussed later.

3.3.10 Symmetry

Let us write the real and imaginary parts of the transform pair $f(t)$ and $F(\nu)$ explicitly as follows:

$$f(t) = f_r(t) + \mathrm{j}f_i(t)$$

$$F(\nu) = F_r(\nu) + \mathrm{j}F_i(\nu)$$

In these, f_r, f_i, F_r and F_i are all **real** functions,

1. $f(t)$ real $\Longrightarrow F(-\nu) = F^*(\nu)$, i.e. F is hermitian (meaning that F_r is even and F_i is odd).
2. $f(t)$ imaginary $\Longrightarrow F(-\nu) = -F^*(\nu)$, i.e. F is antihermitian (meaning that F_r is odd and F_i is even).
3. $f(t)$ even $\Longrightarrow F(-\nu) = F(\nu)$, i.e., $F(\nu)$ is even.
4. $f(t)$ odd $\Longrightarrow F(-\nu) = -F(\nu)$, i.e., $F(\nu)$ is odd.

Some combinations of these cases are also meaningful. For example, the Fourier transform of a real, odd function must be both hermitian and odd, which means that it must be purely imaginary.

Any arbitrary function $f(t)$ can be written as the sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

$$f(t) = f_e(t) + f_o(t) \tag{3.30}$$

$$f_e(t) = \frac{f(t) + f(-t)}{2} \tag{3.31}$$

$$f_o(t) = \frac{f(t) - f(-t)}{2} \tag{3.32}$$

Therefore, if $f(t)$ is real,

$$f_e(t) \leftrightarrow F_r(\nu) \tag{3.33}$$

$$f_o(t) \leftrightarrow \mathrm{j}F_i(\nu) \tag{3.34}$$

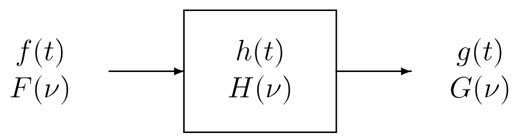
Thus for a **real** function $f(t)$, the real part of the Fourier transform is due to the even part of $f(t)$ and the imaginary part of the transform is due to the odd part of $f(t)$.

For a **causal** real function (i.e., $f(t) = 0$ for $t < 0$), we have in addition that

$$\begin{aligned} f_e(t) = f_o(t) &= \tfrac{1}{2} f(t) \quad \text{if } t > 0 \\ f_e(t) + f_o(t) &= 0 \qquad\quad \text{if } t < 0 \end{aligned} \tag{3.35}$$

This dependence between f_e and f_o means that there is also a dependence between F_r and F_i.

3.3.11 Convolution



Theorem: If $g(t) = (f * h)(t)$ then $G(\nu) = F(\nu)\, H(\nu)$.

Proof:

$$G(\nu) = \int_{-\infty}^{\infty} \exp(-\mathrm{j}2\pi\nu t) g(t)\,\mathrm{d}t \tag{3.36}$$

$$= \int_{-\infty}^{\infty} \exp(-\mathrm{j}2\pi\nu t) \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau)\,\mathrm{d}\tau \right] \mathrm{d}t \tag{3.37}$$

Assuming that we can interchange the order of the integrations,

$$G(\nu) = \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} \exp(-\mathrm{j}2\pi\nu t) h(t-\tau)\,\mathrm{d}t \right] \mathrm{d}\tau \tag{3.38}$$

The term in the brackets is the Fourier transform of $h(t-\tau)$. By the time shifting property,

$$\int_{-\infty}^{\infty} \exp(-\mathrm{j}2\pi\nu t) h(t-\tau)\,\mathrm{d}t = \exp(-\mathrm{j}2\pi\nu\tau) H(\nu) \tag{3.39}$$

so that

$$G(\nu) = \int_{-\infty}^{\infty} f(\tau) \exp(-\mathrm{j}2\pi\nu\tau) H(\nu)\,\mathrm{d}\tau = F(\nu) H(\nu) \tag{3.40}$$

Because of the symmetry in the forward and inverse Fourier transform relationships, we also see that if $f(t) = f_1(t)\, f_2(t)$ then $F(\nu) = (F_1 * F_2)(\nu)$. i.e.,

$$f_1(t)\, f_2(t) \leftrightarrow (F_1 * F_2)(\nu) \tag{3.41}$$

3.3.12 Parseval's theorem

Theorem:

$$\int_{-\infty}^{\infty} f^*(t) g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} F^*(\nu) G(\nu)\,\mathrm{d}\nu \tag{3.42}$$

Proof:

$$\int_{-\infty}^{\infty} f^*(t)g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f^*(t)\left[\int_{-\infty}^{\infty} G(\nu)\exp(\mathrm{j}2\pi\nu t)\right]\mathrm{d}t \tag{3.43}$$

$$= \int_{-\infty}^{\infty} G(\nu)\left[\int_{-\infty}^{\infty} f^*(t)\exp(\mathrm{j}2\pi\nu t)\,\mathrm{d}t\right]\mathrm{d}\nu \tag{3.44}$$

$$= \int_{-\infty}^{\infty} G(\nu)\left[\int_{-\infty}^{\infty} f(t)\exp(-\mathrm{j}2\pi\nu t)\,\mathrm{d}t\right]^*\mathrm{d}\nu \tag{3.45}$$

$$= \int_{-\infty}^{\infty} F^*(\nu)\,G(\nu)\,\mathrm{d}\nu \tag{3.46}$$

The expression $\int_{-\infty}^{\infty} f^*(t)g(t)\,\mathrm{d}t$ can be thought of as an **inner product** of the two functions $f(t)$ and $g(t)$, and is written in Dirac notation as $\langle f(t)|g(t)\rangle$. Parseval's theorem thus states that the inner product is invariant under a Fourier transformation.

$$\langle f(t)|g(t)\rangle = \langle F(\nu)|G(\nu)\rangle \tag{3.47}$$

Note however that there are some definitions of the Fourier transform in which different scalings are used and for which the equality of the inner products has to be replaced by the proportionality of the inner products between the two spaces.

Parseval's theorem can also be written in the alternative forms

$$\int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} F(-\nu)G(\nu)\,\mathrm{d}\nu = \int_{-\infty}^{\infty} F(\nu)G(-\nu)\,\mathrm{d}\nu \tag{3.48}$$

$$\int_{-\infty}^{\infty} f(-t)g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)g(-t)\,\mathrm{d}t = \int_{-\infty}^{\infty} F(\nu)G(\nu)\,\mathrm{d}\nu \tag{3.49}$$

both of which follow from the behaviour of the Fourier transform under conjugation. Using the notation

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t \tag{3.50}$$

just as when we defined the functional induced by a locally integrable function, Parseval's theorem may also be written as

$$\langle f(t), g(t)\rangle = \langle F(-\nu), G(\nu)\rangle = \langle F(\nu), G(-\nu)\rangle \tag{3.51}$$

$$\langle f(-t), g(t)\rangle = \langle f(t), g(-t)\rangle = \langle F(\nu), G(\nu)\rangle \tag{3.52}$$

As we shall see, these form the basis for defining the Fourier transform of generalized functions.

3.3.13 Energy invariance

Putting $f(t) = g(t)$ in Parseval's theorem gives

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |F(\nu)|^2\,\mathrm{d}\nu \tag{3.53}$$

Thus the energy (or its equivalent) is the same in each domain. This result is called **Rayleigh's** theorem.

3.4 Examples of Fourier Transforms

3.4.1 The rectangular pulse

$$f(t) = A\Pi\left(\frac{t}{T}\right) = \begin{cases} A & \text{if } |t| < T/2 \\ 0 & \text{otherwise} \end{cases} \tag{3.54}$$

Using the definition of the Fourier transform

$$F(\nu) = \int_{-T/2}^{T/2} A\exp(-\mathrm{j}2\pi\nu t)\,\mathrm{d}t \tag{3.55}$$

$$= A\left[\frac{\exp(-\mathrm{j}2\pi\nu t)}{-\mathrm{j}2\pi\nu}\right]_{-T/2}^{T/2} \tag{3.56}$$

$$= AT\left(\frac{\sin \pi\nu T}{\pi\nu T}\right) \tag{3.57}$$

$$= AT\,\mathrm{sinc}(\nu T) \tag{3.58}$$

Notice that

1. The zeros of $F(\nu)$ are at integer multiples of $1/T$. Thus the wider is the pulse in the t domain, the narrower is the transform in the ν domain.
2. The value of $F(0)$ is equal to the area under the graph of $f(t)$. Similarly, the value of $f(0)$ is equal to the area under $F(\nu)$. These are general results which follow immediately from the definitions and are useful for checking.

$$F(0) = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t \quad \text{and} \quad f(0) = \int_{-\infty}^{\infty} F(\nu)\,\mathrm{d}\nu \tag{3.59}$$

3.4.2 Two pulses

$$f(t) = \begin{cases} 1 & \text{if } -T < t < 0 \\ -1 & \text{if } 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{3.60}$$

This can be written as

$$f(t) = \Pi\left(\frac{t - \frac{1}{2}T}{T}\right) - \Pi\left(\frac{t + \frac{1}{2}T}{T}\right) \tag{3.61}$$

Using linearity, the shifting theorem and the previous result,

$$F(\nu) = T\exp\left(\mathrm{j}2\pi\nu\frac{T}{2}\right)\mathrm{sinc}(\nu T) - T\exp\left(-\mathrm{j}2\pi\nu\frac{T}{2}\right)\mathrm{sinc}(\nu T) \tag{3.62}$$

$$= \frac{2\mathrm{j}}{\pi\nu}\sin^2(\pi\nu T) \tag{3.63}$$

3.4.3 Triangular pulse

$$f(t) = \begin{cases} \dfrac{T+t}{T} & \text{if } -T < t < 0 \\ \dfrac{T-t}{T} & \text{if } 0 < t < T \\ 0 & \text{otherwise} \end{cases} \tag{3.64}$$

This is T^{-1} multiplied by the integral of the previous example. Since the area under the function in the previous example was zero, we can make use of (3.27) to conclude that

$$F(\nu) = \frac{1}{\mathrm{j}2\pi\nu T}\frac{2\mathrm{j}}{\pi\nu}\sin^2(\pi\nu T) \tag{3.65}$$

$$= T\,\mathrm{sinc}^2(\nu T) \tag{3.66}$$

Alternatively, we see that

$$f(t) = \frac{1}{T}(p * p)(t) \tag{3.67}$$

where $p(t) = \Pi(t/T)$. Using the convolution theorem,

$$F(\nu) = \frac{1}{T}[T\,\mathrm{sinc}(\nu T)]^2 = T\,\mathrm{sinc}^2(\nu T) \tag{3.68}$$

3.4.4 The exponential pulse

$$f(t) = u(t)\exp(-\alpha t) \tag{3.69}$$

Substituting into the definition of the Fourier transform,

$$F(\nu) = \int_0^\infty \exp[(-\alpha - \mathrm{j}2\pi\nu)t]\,\mathrm{d}t \tag{3.70}$$

$$= \frac{1}{\alpha + \mathrm{j}2\pi\nu} \tag{3.71}$$

$$= \frac{\alpha}{\alpha^2 + 4\pi^2\nu^2} - \mathrm{j}\frac{2\pi\nu}{\alpha^2 + 4\pi^2\nu^2} \tag{3.72}$$

$$= \frac{1}{\sqrt{\alpha^2 + 4\pi^2\nu^2}}\exp\left(-\mathrm{j}\tan^{-1}\frac{2\pi\nu}{\alpha}\right) \tag{3.73}$$

3.4.5 The Gaussian

$$f(t) = \exp(-\alpha t^2) \tag{3.74}$$

Substituting into the definition of the Fourier transform,

$$F(\nu) = \int_{-\infty}^\infty \exp[-\alpha t^2 - \mathrm{j}2\pi\nu t]\,\mathrm{d}t \tag{3.75}$$

$$= \int_{-\infty}^\infty \exp\left[-\alpha\left(t^2 + \mathrm{j}\frac{2\pi\nu t}{\alpha}\right)\right]\,\mathrm{d}t \tag{3.76}$$

Completing the square in the exponential,

$$F(\nu) = \exp\left(-\frac{\pi^2\nu^2}{\alpha}\right)\int_{-\infty}^\infty \exp\left[-\alpha\left(t + \mathrm{j}\frac{\pi\nu}{\alpha}\right)^2\right]\,\mathrm{d}t \tag{3.77}$$

$$= \exp\left(-\frac{\pi^2\nu^2}{\alpha}\right)\int_{-\infty+\mathrm{j}\pi\nu/\alpha}^{\infty+\mathrm{j}\pi\nu/\alpha} \exp\left(-\alpha u^2\right)\,\mathrm{d}u \tag{3.78}$$

where we have substituted $u = t + j\pi\nu/\alpha$ in the last integral, so that $du = dt$. It remains to compute the integral which is a contour integral in the complex plane. To do this, consider the integral over the following rectangular contour, where R is later going to be taken to ∞.

$$\int_{-R+j\pi\nu/\alpha}^{R+j\pi\nu/\alpha} + \int_{R+j\pi\nu/\alpha}^{R} + \int_{R}^{-R} + \int_{-R}^{-R+j\pi\nu/\alpha} \exp\left(-\alpha u^2\right) du \tag{3.79}$$

where in each integral, the straight line path between the limits is taken. Since the integrand is analytic throughout the complex plane, Cauchy's theorem states that the integral over any closed contour is zero. As R becomes large, the integrals along the lines $[R + j\pi\nu/\alpha, R]$ and $[-R, -R+j\pi\nu/\alpha]$ become small since the integrand falls off rapidly while the length of the contour stays fixed. Hence in the limit as $R \to \infty$, these integrals vanish and we find that

$$\int_{-\infty+j\pi\nu/\alpha}^{\infty+j\pi\nu/\alpha} \exp\left(-\alpha u^2\right) du + \int_{\infty}^{-\infty} \exp\left(-\alpha u^2\right) du = 0 \tag{3.80}$$

or

$$\int_{-\infty+j\pi\nu/\alpha}^{\infty+j\pi\nu/\alpha} \exp\left(-\alpha u^2\right) du = \int_{-\infty}^{\infty} \exp\left(-\alpha u^2\right) du = \sqrt{\frac{\pi}{\alpha}} \tag{3.81}$$

Thus we see that the desired Fourier transform is

$$F(\nu) = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\pi^2\nu^2}{\alpha}\right) \tag{3.82}$$

A convenient way of remembering this result is to apply the scaling property of Fourier transforms to the special case

$$\exp(-\pi t^2) \leftrightarrow \exp(-\pi\nu^2) \tag{3.83}$$

Exercise: Find the Fourier transforms of the following functions. Sketch the functions and (the real and imaginary parts of) their Fourier transforms.

1. $f(t) = \exp(-\alpha|t|)$
2. $f(t) = \Pi(t/n)\sin(2\pi t)$
3. $f(t) = t^3 \exp(-\alpha t^2)$
4. $f(t) = \exp(-\alpha|t|)\cos(\beta t)$
5. $f(t) = \sum_{k=-N}^{N} \Pi((t-k)/w)$

3.5 Convergence of the Dirichlet Integrals

(See T.M. Apostol, *Mathematical Analysis* for further details.)

In order to show that the inverse Fourier transform formula allows us to recover $f(t)$, we need to consider the Dirichlet integral (3.6)

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} f(\tau) \frac{\sin\left[2\pi M\left(t-\tau\right)\right]}{\pi\left(t-\tau\right)} d\tau \tag{3.84}$$

The inverse Fourier transform at t converges to the value of this integral. Without loss of generality, let us just consider the point $t = 0$. Given $\delta > 0$ we can split the integration into three intervals

$$\lim_{M\to\infty}\int_{-\infty}^{-\delta} f(\tau)\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau + \lim_{M\to\infty}\int_{-\delta}^{\delta} f(\tau)\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau + \lim_{M\to\infty}\int_{\delta}^{\infty} f(\tau)\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau \tag{3.85}$$

In the following, we show that the first and third integrals tend to zero wheras the second integral tends to $f(0)$ if f is sufficiently well behaved.

3.5.1 The Riemann-Lebesgue lemma

Theorem: If f is absolutely integrable on a general interval (a, b) which may be bounded or unbounded,

$$\lim_{M\to\infty}\int_a^b f(t)\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t = 0 \tag{3.86}$$

Proof: It is a result of integration theory that any absolutely integrable function f can be approximated arbitrarily accurately by a step function in the sense that for any $\epsilon > 0$ there is a step function $s(t)$ such that

1. $s(t)$ vanishes outside some bounded interval, and
2. the absolute difference between $s(t)$ and $f(t)$ satisfies

$$\int_a^b |s(t) - f(t)|\,\mathrm{d}t < \epsilon \tag{3.87}$$

(**Note:** A function on (a, b) is a **step function** if there is a partition of (a, b) such that the function is constant on the open subintervals.)

We first show that the Riemann-Lebesgue lemma holds for a constant function on an arbitrary interval and hence also for step functions. The relationship (3.87) is then used to extend the result to all absolutely integrable functions.

1. If $f(t) = c$ is a constant on (a, b),

$$\lim_{M\to\infty}\int_a^b c\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t = \lim_{M\to\infty}\left[c\frac{\exp(\mathrm{j}2\pi Mt)}{\mathrm{j}2\pi M}\right]_a^b = 0 \tag{3.88}$$

   since the numerator is bounded while the denominator becomes large. Note that this proof works whether or not the interval is bounded.

2. If $f(t)$ is a step function, it may be expressed as a sum of functions which are constant on disjoint intervals. The above proof may be applied to each of these disjoint intervals. Since the overall integral is the sum of the integrals over the intervals, the result is true for step functions.

3. Now suppose that $f(t)$ is absolutely integrable and that $s(t)$ is a step function such that (3.87) holds

$$\begin{aligned}\left|\int_a^b f(t)\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t\right| &\leq \left|\int_a^b [f(t)-s(t)]\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t\right| + \left|\int_a^b s(t)\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t\right| \\ &\leq \int_a^b |[f(t)-s(t)]\exp(\mathrm{j}2\pi Mt)|\,\mathrm{d}t + \left|\int_a^b s(t)\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t\right| \\ &= \int_a^b |f(t)-s(t)|\,\mathrm{d}t + \left|\int_a^b s(t)\exp(\mathrm{j}2\pi Mt)\,\mathrm{d}t\right| \qquad (3.89)\end{aligned}$$

By (3.87) the first integral is less than ϵ and by the result for step functions the second integral tends to zero as $M \to \infty$. Since ϵ can be chosen to be arbitrarily small, we have established the required result.

From the Riemann-Lebesgue lemma, it is easy to see that for any absolutely integrable function $f(t)$,

$$\lim_{M\to\infty}\int_a^b f(t)\cos(2\pi Mt)\,\mathrm{d}t = 0 \qquad (3.90)$$

and

$$\lim_{M\to\infty}\int_a^b f(t)\sin(2\pi Mt)\,\mathrm{d}t = 0 \qquad (3.91)$$

In equation (3.85), if $f(\tau)$ is absolutely integrable, then so is $f(\tau)/(\pi\tau)$ on $[\delta,\infty)$ and $(-\infty,-\delta]$. Thus, the Riemann-Lebesgue lemma may be applied to the first and third integrals to show that they tend to zero as $M \to \infty$.

3.5.2 The integral over $(-\delta,\delta)$ in (3.85)

It can be shown that if $f(\tau)$ is continuous and of bounded variation on $(-\delta,\delta)$, the second integral in (3.85) tends to $f(0)$. However, this is quite difficult (see Apostol for details).

(*Technical note:* It is a curious and important result from functional analysis that it is **not** sufficient for f to be continuous at 0, i.e., there exists an absolutely integrable function which is continuous at 0 but whose value at 0 cannot be recovered from an inverse Fourier transform.)

However if $f(\tau)$ is differentiable at 0, the result follows quite readily. We may write

$$\lim_{M\to\infty}\int_{-\delta}^{\delta} f(\tau)\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau = \lim_{M\to\infty}\int_{-\delta}^{\delta}\frac{f(\tau)-f(0)}{\pi\tau}\sin(2\pi M\tau)\,\mathrm{d}\tau + f(0)\lim_{M\to\infty}\int_{-\delta}^{\delta}\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau \qquad (3.92)$$

If $f'(0)$ exists, the first integral on the right-hand side tends to zero as $M \to \infty$ by the Riemann-Lebesgue lemma provided δ is sufficiently small. For this value of δ, the second term is

$$f(0)\lim_{M\to\infty}\int_{-\delta}^{\delta}\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau = \frac{f(0)}{\pi}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,\mathrm{d}x = f(0) \qquad (3.93)$$

This establishes the desired result.

Note:

We have shown that if f is absolutely integrable and differentiable at zero,

$$\lim_{M\to\infty}\int_{-\infty}^{\infty} f(\tau)\frac{\sin(2\pi M\tau)}{\pi\tau}\,\mathrm{d}\tau = f(0) \qquad (3.94)$$

Thus in particular, for any test function ϕ (which certainly satisfies these conditions),

$$\lim_{M\to\infty}\left\langle \frac{\sin(2\pi M\tau)}{\pi\tau}, \phi(\tau)\right\rangle = \phi(0) \tag{3.95}$$

We may write this as a distributional limit

$$\lim_{M\to\infty}\frac{\sin(2\pi M\tau)}{\pi\tau} = \delta(\tau) \tag{3.96}$$

3.6 Fourier Transforms of Generalized Functions

We now wish to extend Fourier transform theory to allow us to find the Fourier transforms of generalized functions or distributions. To do this, it will turn out that we shall always want our test functions to have invertible Fourier transforms (in the classical sense) and that the Fourier transform of a test function should always be another test function. Since the Fourier transform of a function that vanishes outside a bounded interval does not vanish outside a bounded interval, we cannot use exactly the same set of test functions $\mathcal{D}$ that we introduced in the first chapter. Instead we define a new set of test functions (called **open support test functions**) $\mathcal{D}'$ as follows:

A function ϕ is an open support test function if it is infinitely differentiable on the real line (i.e., it is C^∞) and if for all integers $k \geq 0$, the kth derivative of ϕ is **rapidly decreasing**, i.e., for all integers $N \geq 0$,

$$t^N \phi^{(k)}(t) \tag{3.97}$$

is bounded for all t.

This means that **all** derivatives of ϕ tend to zero more quickly than $|t|^{-N}$ for any N as $|t|$ becomes large.

Convergence in $\mathcal{D}'$: A sequence of open support test functions $\{\phi_n(t)\}$ **converges to zero** in $\mathcal{D}'$ if for each pair of non-negative integers N and k, the sequence of functions $t^N\phi_n^{(k)}(t)$ approaches the zero function **uniformly** as $n \to \infty$.

Theorem: The Fourier transform of an open support test function is an open support test function.

Proof: Suppose that $\phi(t)$ is an open support test function.

1. It is easy to show (e.g. by the comparison test with $1/(1+t^2)$) that all rapidly decreasing functions are absolutely integrable and so the Fourier transform $\Phi(\nu)$ exists.

2. In order to show that $\Phi(\nu)$ is infinitely differentiable, we note that $\Phi^{(k)}(\nu)$ is the Fourier transform of $(-\mathrm{j}2\pi t)^k\phi(t)$. Since $\phi(t)$ is rapidly decreasing, $(-\mathrm{j}2\pi t)^k\phi(t)$ is also rapidly decreasing and so it has a Fourier transform.

3. In order to show that $\Phi^{(k)}(\nu)$ is rapidly decreasing, we need to show that for all non-negative natural numbers N, $\nu^N\Phi^{(k)}(\nu)$ is bounded. This is proportional to the Fourier transform of the N'th derivative of $(-\mathrm{j}2\pi t)^k\phi(t)$. Since ϕ is C^∞ and rapidly decreasing, the N'th derivative of $(-\mathrm{j}2\pi t)^k\phi(t)$ is also C^∞ and rapidly decreasing, which means that its Fourier transform is bounded for all ν.

An exactly analogous theory of distributions as in the first chapter can be built up using the class of open support test functions, except that the locally integrable functions are replaced by locally integrable functions which are **slowly increasing**. A function $f(t)$ is said to be slowly increasing if there exists some non-negative integer N such that $f(t)/t^N \to 0$ as $|t| \to \infty$. The distributions induced by this construction are known as **tempered distributions** and they form a subset of the distributions considered in the first chapter.

From this point onwards, a "test function" refers to an open support test function, "generalized functions" and "distributions" refer to tempered distributions.

3.6.1 Definition of the Fourier transform and its inverse

Let $f(t)$ be a generalized function. The Fourier transform $F(\nu)$ is also a generalized function whose action on a test function $\Phi(\nu) \in \mathcal{D}'$ is

$$\langle F(\nu), \Phi(\nu)\rangle = \langle f(t), \phi(-t)\rangle \tag{3.98}$$

where ϕ is the inverse Fourier transform of Φ. By the definition of $\mathcal{D}'$, we know that ϕ is also a test function and so the action of $f(t)$ on $\phi(-t)$ is well-defined.

Similarly, given a generalized function $F(\nu)$, the inverse Fourier transform is a generalized function $f(t)$ whose action on a test function $\phi(t) \in \mathcal{D}'$ is

$$\langle f(t), \phi(t)\rangle = \langle F(\nu), \Phi(-\nu)\rangle \tag{3.99}$$

Theorem: Using the above definitions, if $F(\nu)$ is the Fourier transform of $f(t)$, then $f(t)$ is the inverse Fourier transform of $F(\nu)$.

Proof: Suppose that $F(\nu)$ is the Fourier transform of $f(t)$ and that $g(t)$ is the inverse Fourier transform of $F(\nu)$. Let $\phi(t) \in \mathcal{D}'$ be a test function and $\Phi(\nu)$ be its Fourier transform. Using (3.99), the action of g on ϕ is

$$\langle g(t), \phi(t)\rangle = \langle F(\nu), \Phi(-\nu)\rangle \tag{3.100}$$

Writing $\Psi(\nu) = \Phi(-\nu)$, and using (3.98),

$$\langle F(\nu), \Psi(\nu)\rangle = \langle f(t), \psi(-t)\rangle \tag{3.101}$$

where $\psi(t)$ is the inverse Fourier transform of $\Psi(\nu)$. Using the symmetry properties of the classical Fourier transform on the space of test functions we see that

$$\psi(-t) \leftrightarrow \Psi(-\nu) = \Phi(\nu) \leftrightarrow \phi(t) \tag{3.102}$$

and so $\psi(-t) = \phi(t)$. This shows that $\langle g(t), \phi(t)\rangle = \langle f(t), \phi(t)\rangle$ for all test functions ϕ and so $f = g$ distributionally.

Notes:

1. The above definitions show that **every** generalized function has a Fourier transform which is another generalized function, and that the inverse Fourier transform **always** recovers the original distribution.
2. The duality result that $f(t) \leftrightarrow F(\nu)$ iff $F(t) \leftrightarrow f(-\nu)$ works without exception if we consider generalized functions.
3. The definitions are consistent with (and motivated by) Parseval's theorem and so the classical Fourier transform of a generalized function which is in fact a well-behaved "ordinary" function is also a distributional Fourier transform in the above sense.

3.7 Examples of Fourier transforms of generalized functions

3.7.1 The delta function

If we substitute $f(t) = \delta(t)$ into the definition of the Fourier transform (3.1) and carry out the formal operation of setting $t = 0$ in the exponential (ignoring the fact that $\exp(-\mathrm{j}2\pi\nu t)$ is **not** in the class of test functions $\mathcal{D}'$), we might expect that $F(\nu) = 1$. Let us now see how this is made rigorous through the use of (3.98).

According to the definition, $F(\nu)$ is a generalized function. It is well-defined provided that we can specify its action on a test function $\Phi(\nu)$. Using the definition,

$$\langle F(\nu), \Phi(\nu)\rangle = \langle \delta(t), \phi(-t)\rangle \tag{3.103}$$

$$= \phi(0) = \int_{-\infty}^{\infty} \Phi(\nu)\,\mathrm{d}\nu = \langle 1, \Phi(\nu)\rangle \tag{3.104}$$

Since this is true for all $\Phi \in \mathcal{D}'$, $F(\nu) = 1$ distributionally.

Similarly, if $f(t) = \delta(t - T)$,

$$\langle F(\nu), \Phi(\nu)\rangle = \langle \delta(t-T), \phi(-t)\rangle \tag{3.105}$$

$$= \phi(-T) = \int_{-\infty}^{\infty} \Phi(\nu)\exp[\mathrm{j}2\pi\nu(-T)]\,\mathrm{d}\nu = \langle \exp(-\mathrm{j}2\pi\nu T), \Phi(\nu)\rangle \tag{3.106}$$

Hence we may write the Fourier transform pair

$$\delta(t - T) \leftrightarrow \exp(-\mathrm{j}2\pi\nu T) \tag{3.107}$$

Exercise: Show using (3.98) that if $g(t) = f(t - T)$ then $G(\nu) = F(\nu)\exp(-\mathrm{j}2\pi\nu T)$ so that the above is a special case of this relationship.

3.7.2 The complex exponential

By duality we expect the Fourier transform of $f(t) = \exp(\mathrm{j}2\pi\nu_0 t)$ to be $\delta(\nu - \nu_0)$. We can also see this directly since for any test function Φ,

$$\langle F(\nu), \Phi(\nu)\rangle = \langle \exp(\mathrm{j}2\pi\nu_0 t), \phi(-t)\rangle \tag{3.108}$$

$$= \int_{-\infty}^{\infty} \exp(\mathrm{j}2\pi\nu_0 t)\phi(-t)\,\mathrm{d}t = \Phi(\nu_0) = \langle \delta(\nu - \nu_0), \Phi(\nu)\rangle \tag{3.109}$$

In this case, if we had formally substituted $f(t)$ into the original definition of the Fourier transform, we would have obtained

$$F(\nu) = \int_{-\infty}^{\infty} \exp(\mathrm{j}2\pi(\nu_0 - \nu)t)\,\mathrm{d}t$$

This improper integral does not have a limit in the usual sense. However, on the basis of the above, we may formally write

$$\int_{-\infty}^{\infty} \exp(\mathrm{j}2\pi(\nu_0 - \nu)t)\,\mathrm{d}t = \delta(\nu - \nu_0)$$

provided that this is understood in the distributional sense.

3.7.3 Cosine and sine

By writing the cosine and sine functions in terms of complex exponentials we see that

1. $\cos(2\pi\nu_0 t) \leftrightarrow \frac{1}{2}[\delta(\nu-\nu_0)+\delta(\nu+\nu_0)]$
2. $\sin(2\pi\nu_0 t) \leftrightarrow \frac{1}{2\mathrm{j}}[\delta(\nu-\nu_0)-\delta(\nu+\nu_0)]$

Note that the cosine function is real and even and so its Fourier transform is Hermitian and even (and consequently real). On the other hand, the sine function is real and odd and so its Fourier transform is Hermitian and odd (and consequently purely imaginary).

3.7.4 The signum function

This is defined by

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases} \tag{3.110}$$

One way of calculating the Fourier transform of this function is by considering it as the (distributional) limit of a sequence of functions. This relies on the following theorem:

Theorem: If $\{f_k\}$ is a convergent sequence of generalized functions which converge to the generalized function f, then the sequence of Fourier transforms $\{F_k\}$ converge distributionally to the Fourier transform F of f.

Proof: Do as an exercise using the definitions of Fourier transforms and limits of generalized functions.

Consider the sequence of functions

$$f_k(t) = \begin{cases} \exp(-t/k) & \text{if } t > 0 \\ -\exp(t/k) & \text{if } t < 0 \end{cases} \tag{3.111}$$

As $k \to \infty$ it is easy to see that this tends to $\operatorname{sgn} t$ pointwise and distributionally. For each k, $f_k(t)$ is absolutely integrable and so its Fourier transform exists as an ordinary function

$$F_k(\nu) = \frac{1}{\mathrm{j}2\pi\nu + k^{-1}} - \frac{1}{\mathrm{j}2\pi(-\nu) + k^{-1}} \tag{3.112}$$

$$= -\frac{\mathrm{j}4\pi\nu}{k^{-2} + 4\pi^2\nu^2} \tag{3.113}$$

As $k \to \infty$ this tends to $1/(\mathrm{j}\pi\nu)$. Thus we have the transform pair

$$\operatorname{sgn} t \leftrightarrow \frac{1}{\mathrm{j}\pi\nu} \tag{3.114}$$

3.7.5 The unit step – Fourier transform of an integral

Since the unit step $u(t)$ may be written in terms of the signum function

$$u(t) = \frac{1}{2}(\operatorname{sgn}(t) + 1) \tag{3.115}$$

we may use linearity and the previous result to obtain the Fourier transform pair

$$u(t) \leftrightarrow \frac{1}{j2\pi\nu} + \frac{1}{2}\,\delta(\nu) \tag{3.116}$$

Exercise: Consider the sequence of functions $f_k(t) = u(t)\exp(-t/k)$ which tends distributionally to $u(t)$ as $k \to \infty$. Calculate $F_k(\nu)$ and show that these tend distributionally to the Fourier transform of $u(t)$. **Hint:** Consider the real and imaginary parts of $F_k(\nu)$ separately.
(*Technical note:* This shows that pointwise convergence does not imply distributional convergence.)

Recall that given a function $f(t)$, the integral function

$$g(t) = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau \tag{3.117}$$

can be written as the convolution $(f * u)(t)$. By the convolution theorem and the above result, the Fourier transform of $g(t)$ is

$$G(\nu) = F(\nu)\left(\frac{1}{j2\pi\nu} + \frac{1}{2}\,\delta(\nu)\right) = \frac{F(\nu)}{j2\pi\nu} + \frac{1}{2}F(0)\delta(\nu) \tag{3.118}$$

This is the promised generalization of (3.27) which is required when $F(0)$ is non-zero.

Exercise: What are the Fourier transforms of $\operatorname{sgn} t\cos(2\pi\nu_0 t)$, $u(t)\sin(2\pi\nu_0 t)$ and of t^k?