

Sound Waves

• Conservation of momentum

No external forces \rightarrow only internal stress

No viscous stress

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = - \nabla p$$

• Compressibility of Fluid - Conservation of Mass $\frac{\text{divergence}}{+ \rho \nabla \cdot \vec{u}}$

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho \nabla \cdot \vec{u} = 0$$

$$\frac{D\rho}{Dt} = 0$$

• Linearized equations

(ρ_i, u_i) scaled to be $O(1)$

$$\rho = \rho_0 + \rho_1$$

$$\rho_1 \ll \rho_0$$

$$\rho = \rho_0 + \epsilon \rho_1 + O(\epsilon^2)$$

$$u = \epsilon u_0 + O(\epsilon^2)$$

or better $\epsilon \ll 1$

parameter depends on scales used for (ρ_i, u_i, \vec{u}_i)

$$p = p_0 + \epsilon p_1 + O(\epsilon^2)$$

$$O(1): \quad \sigma = - \nabla p_0$$

uniform pressure $p_0 = \text{const. in space}$

$$\frac{\partial \rho_0}{\partial t} = \sigma$$

uniform density $\rho_0 = \text{const. in time}$

$O(\epsilon):$

$$\rho_0 \frac{\partial \vec{u}_0}{\partial t} = - \nabla p_1$$

$$\text{or } \rho_0 \frac{\partial \vec{u}}{\partial t} = - \nabla p$$

$$\frac{\partial \rho_1}{\partial t} + \vec{u}_0 \cdot \nabla \rho_0 = \sigma$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0$$

Set $\vec{u} = \nabla \phi$ Velocity Potential

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Momentum $\rho_0 \frac{\partial}{\partial t} \nabla \phi = -\nabla p$ or $-\rho_0 \frac{\partial \phi}{\partial t} = p - p_0$ (1)

Compressibility $\frac{\partial \rho}{\partial t} + \rho_0 \nabla^2 \phi = 0$ or $\frac{\partial \rho}{\partial t} = -\rho_0 \nabla^2 \phi$ (2)

Need relation between ρ and p (Equation of State)

Assume $p = p(\rho)$ and do Taylor Expansion about ρ_0

$$p = \underbrace{p(\rho_0)}_{\text{steady}} + (\rho - \rho_0) \underbrace{\frac{\partial p}{\partial \rho}}_{\text{steady}} \Big|_{\rho_0} + \dots$$
 (3)

↓

$$\frac{\partial p}{\partial t} = 0 + \frac{\partial \rho}{\partial t} \cdot \underbrace{\frac{\partial p}{\partial \rho} \Big|_{\rho_0}}_{\equiv c^2} = c^2 \frac{\partial \rho}{\partial t}$$

or

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2} \frac{\partial p}{\partial t} = -\rho_0 \nabla^2 \phi \quad \text{using (2)}$$

or

$$-\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \cdot \rho_0 = -\rho_0 \nabla^2 \phi \quad \text{using (1)}$$

↓

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \quad \text{WAVE EQUATION}$$

$$\ddot{\phi} - c^2 \phi'' = 0 \quad \text{Wave equation}$$

Solutions are plane waves
traveling in positive x -direction

$$\phi = f(x-ct)$$

Wave is longitudinal in that

$$u = \frac{\partial \phi}{\partial x}$$

$$\vec{u} = (u, v, w) \quad \text{etc}$$

satisfies $u = \frac{\partial f(x-ct)}{\partial x} \quad v = w = 0$

and pressure

$$p - p_0 = -\rho_0 \frac{\partial \phi}{\partial t} = +\rho_0 c u$$

$$p(x,t) = p_0 + \rho c u(x,t)$$

where ρ excess press \propto fluid velocity

$\phi = g(x+ct)$ is solution also, thus

$$u = \frac{\partial g}{\partial x} \quad v = w = 0$$

and $p - p_0 = -\rho_0 c u$

More generally

$$\phi = h(\xi x + \eta y + \zeta z - ct)$$

with $\xi^2 + \eta^2 + \zeta^2 = 1$

and

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \cdot \frac{1}{\rho c} (p - p_0)$$

- wave phase speed c is the same in all directions
- the shape of the wave form h does not vary as the wave form propagates through the media of density ρ_0 .
- Note that this shape h does NOT have to be sinusoidal.

Recall Speed-of-Sound that was introduced as

$$c^2 = \left. \frac{\partial p}{\partial \rho} \right|_{p_0}$$

What if we have an explicit constitutive law (perfect gas, say) that relates pressure and density, eg.

$$\gamma = \frac{c_p}{c_v} = \frac{\text{specific heat at const. } p}{\text{specific heat at const } V}$$

$$p = \gamma R \cdot T \cdot \rho$$

Then

$$c = \sqrt{\gamma R T}$$

depends on temperature (absolute K) and

$$\gamma R = \frac{8314 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}}{\text{mean molecular weight}} \cdot \gamma = \frac{8314 \cdot \gamma}{29} = 286.7 \cdot \gamma$$

$$c = \sqrt{\frac{286.7 \text{ m}^2}{\text{s}^2 \text{ K}} \cdot 293 \text{ K}} = 290 \frac{\text{m}}{\text{s}} \quad \text{observed is } 340 \frac{\text{m}}{\text{s}}$$