Surface Gravity Waves

- inviscid, incompressible, homogeneous fluid
- free surface "near" $z=0$
- flat bottom $z=-D$

(1) inviscid (no friction)

momentum
$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\,k_z$$

Define vorticity
$$\vec{\omega} = \nabla \times \vec{u}$$

Take curl ($\nabla \times$) of momentum

$$\nabla \times \frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla \times \nabla p - g\,\nabla \times k_z$$

$$\frac{D\vec{\omega}}{Dt} = 0$$

$\vec{\omega}(\vec{x}, t=0) = 0$

$$\cancel{\frac{\partial \vec{\omega}}{\partial t} + \vec{u}\cdot\nabla\vec{\omega} = 0}$$

then $\vec{\omega}(\vec{x}, t) = 0$

$\Rightarrow$ $\vec{\omega} = \nabla \times \vec{u} = 0$ irrotational flow

$\Rightarrow$ $\vec{u} = \nabla \phi$ velocity potential

(2) incompressible $\nabla \cdot \vec{u} = 0$ $\Rightarrow$ $\nabla^2 \phi = 0$

Boundary Conditions

at bottom $z=-D$ $\quad w=0 \quad \longrightarrow \phi_z = 0$ at $z=-D$

at free surface

$$z = \eta(x,y,t)$$

the particles "making up" this surface move

(i) vertical if the interface rises or falls

(ii) horizontal if the surface slopes

$\Downarrow$

$$w(x,y,\eta(x,y,t),t) = \eta_t + u\,\eta_x + v\,\eta_y \quad \text{at } z=\eta$$

which states that $\dfrac{D\eta}{Dt} = w$

or

$$\eta_t + \overset{u}{\phi_x}\,\eta_x + \overset{v}{\phi_y}\,\eta_y = \overset{w}{\phi_z} \quad \text{at } z=\eta$$

This is a kinematic condition that defines our interface

The interface has no mass and without surface tension there are no pressure gradients across it. Thus the dynamical boundary condition is

$$p(x,y,\eta,t) = p_{\text{atmosphere}}$$