

We need to write this in terms of  $\phi$  and  $y$

Lets look at momentum

$$\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla p}{\rho} - g \cdot \vec{k}_2$$

With

$$(\vec{u} \cdot \nabla) \vec{u} = (\underbrace{\nabla \times \vec{u}}_{\vec{\omega}}) \times \vec{u} + \nabla (\vec{u} \cdot \vec{u} / 2)$$

The momentum becomes

$$\vec{u}_t + \vec{\omega} \times \vec{u} = -\frac{\nabla p}{\rho} - \nabla (\vec{u} \cdot \vec{u} / 2) - \nabla g z$$

$\vec{\omega} = 0$  irrotational flow  
to allow  $\vec{u} = \nabla \phi$

$$\nabla \left( \phi_t + \frac{p}{\rho} + \frac{1}{2} |\nabla \phi|^2 + gz \right) = 0$$

or

$$\phi_t + \frac{p}{\rho} + gz + \frac{1}{2} |\nabla \phi|^2 = f(t)$$

This is the so-called Bernoulli integral. Apply at  $z=y$

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + gy = f(t) - \frac{p_{atm}}{\rho}$$

Choose  $f(t)$  to cancel the time-dependent part  
of atmospheric pressure  $p_{atm} = p_{atm}(x, y, t)$

For a spatially uniform atmospheric pressure  $p_{atm} = p_{atm}(t)$   
this becomes

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + g_y = 0 \quad \text{at } z=y$$

Full problem then is

$$\gamma_z + \phi_x \gamma_x + \phi_y \gamma_y = \phi_z \quad \text{at } z=y$$

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + g_y = 0 \quad \text{at } z=y$$

$$\nabla^2 \phi = 0$$

$$\phi_z = 0 \quad \text{at } z=-D$$

It is obvious that the surface boundary conditions  
at  $z=y$ , both kinematic AND dynamic,  
will require some work

Lets consider 1-D problem (1 horizontal dimension)

and drop all non-linear terms (to be checked later)

$$(1) \quad \eta_L = \phi_2 \quad \text{at } z=L/2$$

$$(2) \quad \phi_L + g\eta = 0 \quad \text{at } z=L/2$$

$$(3) \quad \nabla^2 \phi = 0$$

$$(4) \quad \phi_2 = 0 \quad \text{at } z=-D$$

Lets try plane waves solutions

$$\eta = a e^{-i\omega t + ikx}$$

$$\phi = A e^{-i\omega t + ikx} \cdot Z(z)$$

Insert into (3)

$$-k^2 Z + Z_{zz} = 0$$

which gives  $Z(z) = \cosh k(z+D) e^{\pm kz}$

the linear combination of such solutions that satisfies the bottom boundary condition (4) is

$$Z(z) = \cosh [k_r (z+D)] = \frac{e^{+k_r(z+D)}}{2} - \frac{e^{-k_r(z+D)}}{2}$$

The two free surface conditions (1) and (2)  
combine to

$$\phi_{tt} + g \phi_2 = 0 \quad \frac{\partial (2)}{\partial z} \text{ using (1)}$$

at  $z=0$

Insert

$$\phi = A e^{-i\omega t + ikx} \cdot \cosh k(z+D)$$

to find dispersion relation

$$\boxed{\omega^2 = g k \tanh(kD)}$$

The amplitude  $A$  of our velocity potential  $\phi$  relates to  
the amplitude  $a$  of our surface  $y$  from  $y_L = \phi_2$   
and  $\phi_t + ga = 0$

$$A = -\frac{i a \omega}{k \sinh(kD)} = -\frac{i a g}{\omega \cosh(kD)}$$

These are plane waves that are dispersive and  
can propagate in both "+" and "-" directions

For completeness, take the real part of these wave solutions

$$\eta = a \cos(kx - \sigma t)$$

$$\phi = \frac{a\sigma}{k \sinh(kD)} \cosh k(z+D) \sin(kx - \sigma t)$$

$$u = \frac{a\sigma}{\sinh(kD)} \cosh k(z+D) \cos(kx - \sigma t)$$

$$w = \frac{a\sigma}{\sinh(kD)} \sinh k(z+D) \sin(kx - \sigma t)$$

$$p = \frac{\rho\sigma^2 a}{k \sinh(kD)} \cosh k(z+D) \cos(kx - \sigma t) - \rho g z$$

Notice that  $\eta$ ,  $u$ , and  $p$  are in-phase

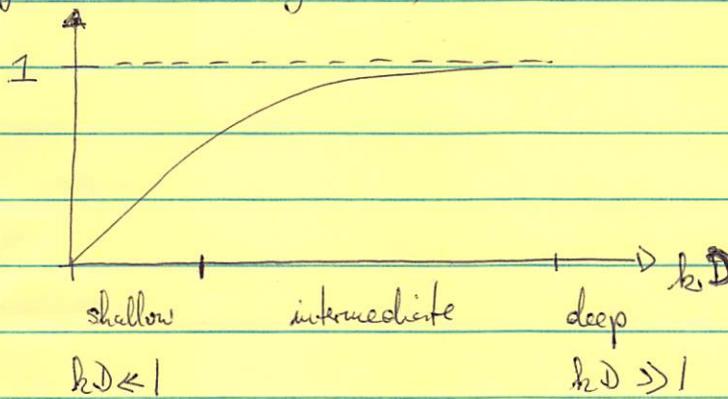
This general derivation applies equally for

deep water waves with  $\lambda \ll D$  or  $kD \gg 1$   
(short waves)

and

shallow water waves with  $\lambda \gg D$  or  $kD \ll 1$   
(long waves)

~~tanh(kD) ≈ 2~~



$$\text{For } kD \ll 1 \quad \tanh kD \approx kD \quad \text{for } kD \gg 1 \quad \tanh kD \approx 1$$

Hence dispersion

$$\omega^2 = g h \tanh kD$$

gives

$$c_p^2 = \frac{\omega^2}{k^2} = \frac{g}{h} \tanh kD \approx \begin{cases} gD & kD \ll 1 \quad \text{shallow water} \\ g/k & kD \gg 1 \quad \text{deep water} \end{cases}$$

So generally, gravity waves are dispersive  $c_p = c_p(k)$

except in the shallow water limit  $c_p = \sqrt{gD}$