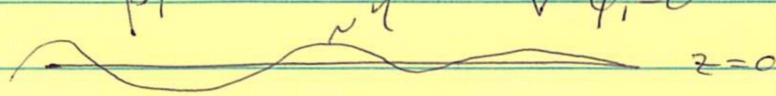
3.3 Internal Waves

Air – Atmosphere not the only interface that can support gravity waves

Consider interface between two fluids of semi-infinite ~~ext~~ depth

ρ_1 $\qquad\qquad$ $\nabla^2 \phi_1 = 0$



ρ_2 $\qquad\qquad$ $\nabla^2 \phi_2 = 0$

At interface $z=0$ we have

kinematic BC $\qquad \eta_t = \phi_{1z}$ and $\eta_t = \phi_{2z}$

dynamic BC (pressure matching) $\qquad \rho_1(\phi_{1t} + g\eta) = \rho_2(\phi_{2t} + g\eta)$

These equations are satisfied with

$$\phi_1 = A_1 e^{-i\sigma t + ikx - kz}$$

$$\phi_2 = A_2 e^{-i\sigma t + ikx + kz}$$

deep water simplifies z-dependence

$$\eta = a\, e^{-i\sigma t + ikx}$$

And the three surface conditions with these solutions become

$$-i\sigma a = -k A_1$$

$$-i\sigma a = +k A_2$$

$$\rho_1\left(\frac{\sigma}{k} + g\right) = \rho_2\left(-\frac{\sigma}{k} + g\right) \qquad \cancel{\text{or } \sigma^2 = g}$$

The pressure matching conditions can be written as

$$\sigma^2 = g k \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

which for $\rho_1 = 0$

$$\sigma^2 = g k$$

which is the deep water dispersion relation

$$g \longrightarrow g \cdot \Delta\rho / \bar{\rho} \approx 10^{-3} g \qquad \begin{aligned} \Delta\rho &= \rho_2 - \rho_1 \\ \bar{\rho} &= \rho_2 + \rho_1 \end{aligned}$$

Much more on internal waves later

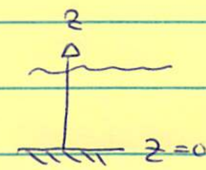
Lets look more closely at the nonlinear governing equations and boundary conditions

(1) $\phi_t + g\eta + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) = 0$ @ $z = D+\eta$

(2) $\eta_t + \phi_x \eta_x + \phi_y \eta_y = \phi_z$ @ $z = D+\eta$

(3) $\nabla^2 \phi = 0$

(4) $\phi_z = 0$ @ $z = 0$ which is the bottom



Introduce non-dimensional variables

$$(x,y) = (x,y)\cdot L$$

$$z = z \cdot D$$

$$t = t \cdot L/\sqrt{gD}$$

$$\eta = \eta \cdot a$$

$$\phi = \phi \cdot gaL/\sqrt{gD}$$

Then dimensional

(2) $$\eta_t + \phi_x \eta_x = \phi_z$$

becomes

$$\frac{a\sqrt{gD}}{L}\eta_t + \frac{gaL}{\sqrt{gD}}\cdot\frac{a}{L^2}\phi_x\eta_x = \frac{gaL}{D\sqrt{gD}}\phi_z$$

or

$$\eta_t + \underbrace{\left(\frac{a}{D}\right)}_{\equiv \varepsilon} \phi_x \eta_x = \underbrace{\left(\frac{L}{D}\right)}_{\equiv \delta}^2 \phi_z \quad @ \quad z = 1 + \left(\frac{a}{D}\right) z$$

$\frac{\text{surface amplitude}}{\text{depth}}$ $\qquad$ aspect ratio $\qquad$ non-dimensional parameters

With these, the other equations write as

(1) $\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \phi_y^2\right) + \frac{\varepsilon}{2\delta^2}\phi_z^2 + \eta = 0 \quad @ \quad z = 1 + \varepsilon\eta$

(2) $\eta_t + \varepsilon\left(\eta_x \phi_x + \phi_y \eta_y\right) = \frac{1}{\delta^2}\phi_z \quad @ \quad z = 1 + \varepsilon\eta$

(3) $\phi_{zz} + \delta^2\left(\phi_{xx} + \phi_{yy}\right) = 0$

(4) $\phi_z = 0 \quad @ \quad z = 0$

$\boxed{\text{Case A :}}$ $\quad \varepsilon \ll 1$ and $\delta = 1$

Li... ar Deep Water Waves

$$\begin{cases} (1)+(2) & \phi_t + \eta = 0 \qquad \eta_t = \phi_z \quad @ \quad z = 1 \\ (3) & \nabla^2 \phi = 0 \\ (4) & \phi_z = 0 \quad @ \quad z = 0 \end{cases}$$