

[1.4]

Method of stationary phase

p.-7

to approximate initial conditions

Example

$$\eta(x, t=0) = a(x) e^{ik_0 x}$$

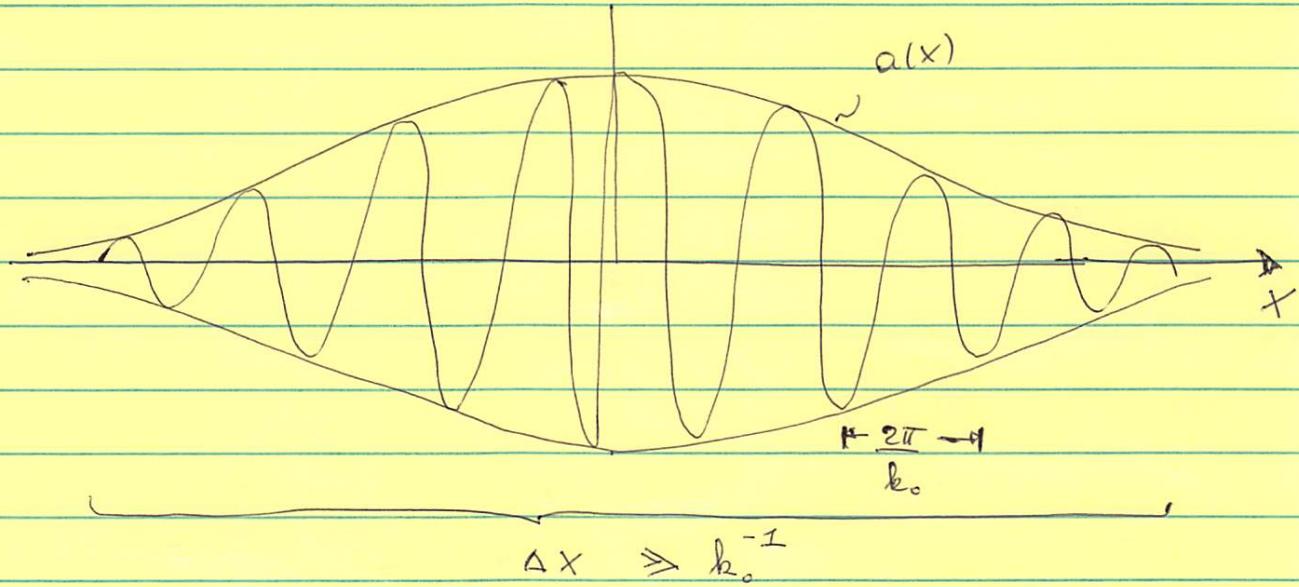
represents a slowly modulated plane wave with envelope $a(x)$

$$\eta(x, \sigma) = \int_{-\infty}^{+\infty} A(k) e^{ikx} dk$$

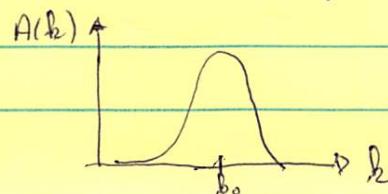
$$A(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta(x, \sigma) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} a(x) e^{i(k_0 - k)x} dx$$

$$a(x) = \int_{-\infty}^{+\infty} A(k) e^{i(k_0 - k)x} dk$$

most contributions come from $(k_0 - k) \cdot x$ small



$(k_0 - k)x$ large \rightarrow rapid oscillations that cancel out
 $(k_0 - k)x$ small stationary phase where signal is slowly varying



the modulated plane wave is said to be "narrow-band"

Then evaluate $\eta(x, t)$ by expanding the frequency $\sigma(k)$

in a Taylor Series about wavenumber k_0 :

$$\begin{aligned} \eta(x, t) &= \int_{-\infty}^{+\infty} A(k) e^{i(kx - \sigma(k)t)} dk \\ &\approx \int_{-\infty}^{+\infty} A(k) e^{i\left(kx - \sigma(k_0)t - (k-k_0)\frac{\partial \sigma}{\partial k}\Big|_{k=k_0} \cdot t\right)} dk \\ &= e^{i(k_0 x - \sigma(k_0)t)} \underbrace{\int_{-\infty}^{+\infty} A(k) e^{i(k-k_0)\left(x - \frac{\partial \sigma}{\partial k}\Big|_{k=k_0} \cdot t\right)} dk}_{e^{i k_0 x - i \frac{\partial \sigma}{\partial k}|_{k=k_0} t} = 1} \end{aligned}$$

$$\eta(x, t) = e^{i(k_0 x - \sigma(k_0)t)} \cdot a\left(x - \frac{\partial \sigma}{\partial k}\Big|_{k=k_0} t\right)$$

The modulating envelope moves at a velocity $\frac{\partial \sigma}{\partial k}\Big|_{k=k_0}$

defined by the dispersion relation $\sigma = \sigma(k)$

This velocity is called the group velocity

$$c_g = \frac{\partial \sigma}{\partial k}\Big|_{k=k_0} \neq \frac{\sigma}{k} = c_p$$

The restriction to "narrow band processes" is not necessary
A theorem exists (Riemann-Lebesgue Theorem) that

If $\int_{-\infty}^{+\infty} A(k) dk$ exists

Then $\lim_{t \rightarrow \infty} \int_{-\infty}^{+\infty} A(k) e^{ikt} dk = 0$

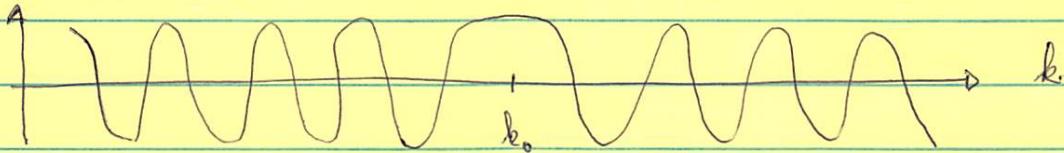
Thus we get little contribution to $y(x, t)$ unless the

phase: $\Theta = \frac{kx}{t} - \sigma(k)$ varies NOT with k

↑

$$\left. \frac{\partial \Theta}{\partial k} \right|_{k=k_0} = 0$$

$A(k) e^{i\Theta}$



rapid oscillation that "cancel"

Stationary Phase now asserts

$$y(x, t) \approx \int_{-\infty}^{+\infty} A(k) e^{it(\Theta(k_0) + (k-k_0)\Theta'(k_0) + \frac{(k-k_0)^2}{2}\Theta''(k_0))} dk$$

but $\Theta = \frac{kx}{t} - \sigma(k)$

$$\Theta' = \frac{d\Theta}{dk} = \frac{x}{t} - \frac{\partial \sigma}{\partial k} \Big|_{k=k_0}$$

$\Theta'' = 0$ gives $\frac{x}{t} = \frac{\partial \sigma}{\partial k} \Big|_{k=k_0}$

The k_0 that makes the largest contribution to $\eta(x, t)$

is the plane wave whose group velocity is x/t

which occurs for $\Theta'(k_0) = 0$

$$\text{A} \quad \eta(x, t) \approx A(k_0) e^{it\Theta(k_0)} \int_{-\infty}^{+\infty} e^{i(k-k_0)^2} \Theta''(k) L/2 d.k$$

$$\text{or since } \int_{-\infty}^{+\infty} e^{-\alpha z^2} dz = (\pi/\alpha)^{1/2}$$

$$\eta(x, t) \approx A(k_0) e^{it\Theta(k_0)} [2\pi/i t \Theta''(k_0)]^{1/2}$$

Solution is valid only for large t and x , because it requires rapid oscillations of $e^{i(kx - \sigma t)}$ at all k except those where

$$x - \frac{\partial \Sigma}{\partial k} t = 0$$

It thus describes waves far from and long after their initial generation

[3.6]

Example: Deep Water Waves

p. 51

$$\sigma = \sigma(k) = (g/k)^{1/2}$$

$$c_p = \frac{\sigma}{k} = (g/k)^{1/2}$$

$$c_g = \frac{\partial \sigma}{\partial k} = \frac{1}{2} (g/k)^{-1/2} \cdot g = \frac{1}{2} \left(\frac{g^2}{g/k} \right)^{1/2} = \frac{1}{2} \left(\frac{g}{k} \right)^{1/2}$$

Stationary Phase

$$\Theta_- = \frac{kx - \sigma(k)}{t}$$

$$\Theta_+ = \frac{kx + \sigma(k)}{t}$$

$$\frac{\partial \Theta_-}{\partial k} - \sigma = \frac{x}{t} - \frac{1}{2} (g/k)^{1/2}$$

no real roots for $\frac{\partial \Theta_+}{\partial k}$

$$\therefore \frac{x}{t} = + \frac{1}{2} (g/k)^{1/2}$$

This reveals k_c ?

$$\therefore \frac{x^3}{t^3} = \frac{1}{8} \frac{g^{3/2}}{k_c^{3/2}} \quad \therefore k_c^{3/2} = \frac{g^{3/2}}{\frac{1}{8} t^3}$$

$$\frac{\partial^2 \Theta_-}{\partial k_c^2} = \frac{1}{4} \frac{g^{1/2}}{k_c^{3/2}} \quad \therefore \frac{1}{4} g^{1/2} \frac{8x^3}{g^{3/2} \cdot t^3}$$

$$= \frac{2x^3}{gt^3}$$

$$\eta(x > 0, t) \approx \frac{1}{2} \tilde{\gamma}_0(k_c) e^{i\Theta(k_c)t} \left[\frac{2\pi}{it\Theta(k_c)} \right]^{1/2}$$

$$\frac{1}{2} \tilde{\gamma}_0(k_c) e$$

$$\sigma = (g \cdot k)^{1/2}$$

$$\Theta = \frac{kx}{t} - (gk)^{1/2}$$

$$\frac{x}{t} = \frac{1}{2} \left(\frac{g}{k_0} \right)^{1/2} \quad ; \quad \frac{x^2}{t^2} = \frac{1}{4} \cdot \frac{g}{k_0} \quad ; \quad k_0 = \frac{g}{4} \cdot \frac{t^2}{x^2}$$

$$\Theta(k_0) = \frac{\frac{g}{4} \frac{t^2}{x^2} \cdot \frac{x}{t}}{t} - \left(g \frac{\frac{g}{4} \frac{t^2}{x^2}}{t} \right)^{1/2}$$

$$= \frac{\frac{g}{4} \frac{t}{x}}{t} - \frac{2g}{2 \cdot 2} \frac{t}{x} = - \frac{gt}{4x}$$

$$i\Theta(k_0) \cdot t = -i \frac{gt^2}{4x}$$

and

$$y(x>0, t) \approx \frac{1}{2} \overline{y}_0(k_0) e^{-\frac{igt^2}{4x}} \cdot e^{-i\pi/4} \left(\frac{\pi gt^2}{x^3} \right)^{1/2}$$

$$\left(\frac{2\pi}{it\Theta} \right)^{1/2} = \left(-2\pi i \frac{gt^3}{2x^3 \cdot t} \right)^{1/2} = \left(-i\pi \right)^{1/2} \left(\frac{gt^2}{2x^3} \right)^{1/2} = e^{-i\pi/4} \left(\frac{\pi gt^2}{x^3} \right)^{1/2}$$

Identical consideration for $k < 0$

Add solutions to obtain

$$\eta(x, t) \approx \bar{\gamma}_0(k_0) e^{-i g t^2 / 4x} (\pi g)^{1/2} \frac{t}{x^{3/2}} e^{-i \pi/4}$$

$$\operatorname{Re}(\eta(x, t)) \approx \gamma_0(k_0) (\pi g)^{1/2} \frac{t}{x^{3/2}} \cos(g t^2 / 4x + \pi/4)$$

$$\text{if } \gamma_0(x, 0) = \delta(x) \quad \text{then} \quad \bar{\gamma}_0(k_0) = 1/(2\pi)$$

20+

$$x = 50 * 0.2$$

$$t = 20 * 0.05$$

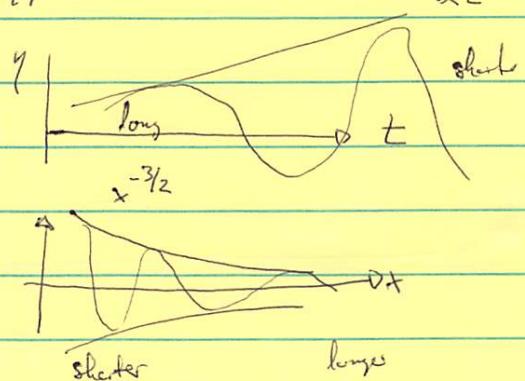
20+

Show solution as damped

$$\theta = \theta(x, t) \quad \text{using GHT}$$

$$\eta = \eta(x = \text{const}, t)$$

$$\eta = \eta(x, t = \text{const})$$



Observations:

- Neither k_0 nor τ_c is constant at any fixed x or t

$$\cdot \tau_{0t} + \left. \frac{\partial \tau}{\partial k} \right|_{k=k_0} \cdot \tau_{0x} = 0$$

$$\frac{D(\tau_0)}{Dt} = 0$$

$$\cdot k_{0t} + \left. \frac{\partial \tau}{\partial k} \right|_{k=k_0} k_{0x} = 0$$

$$\frac{D(k_0)}{Dt} = 0$$

$C_g(k=k_0)$ group velocity