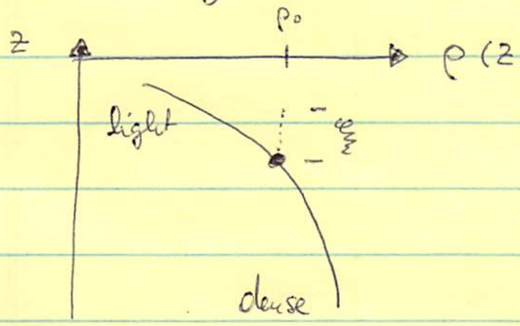
Gravity Waves in Continuously Stratified Fluids



raise particle distance ξ in fluid with $\rho = \rho(z)$

change in pressure

$$dp = -\rho_0 g \xi$$

change in density

$$d\rho = dp / c^2$$

This produces an acceleration due to

buoyancy force ~~buoyancy force~~ $g(\rho_{out} - \rho_{in})$

$$\underbrace{g\,(\rho_{out} - \rho_{in}) = g\left[\left(\rho_0 + \frac{\partial \rho_0}{\partial z}\cdot \xi\right) - \left(\rho_0 - \rho_0 g \frac{\xi}{c^2}\right)\right]}_{\text{buoyancy force}} = \underbrace{\rho_0 \frac{\partial^2 \xi}{\partial t^2}}_{\text{acceleration}}$$

$$\xi_{tt} + \xi \underbrace{\left(\frac{-g}{\rho_0}\frac{\partial \rho_0}{\partial z} - \frac{g^2}{c^2}\right)}_{} = 0$$

Harmonic Oscillation

$$N^2(z) = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} - \underbrace{\frac{g^2}{c^2}}_{\text{compressibility}}$$

Buoyancy frequency (Brunt-Väisälä frequency)

neglect in ocean but NOT in air

Equations of motion (rotating + incompressible fluid)

(1) (2) (3) $$\frac{D\vec{u}^*}{Dt} + f\vec{k} \times \vec{u}^* = -\frac{\nabla \cdot p^*}{\rho^*} - g\vec{k}$$ momentum

(4) $$\frac{\partial \rho^*}{\partial t} + \nabla \cdot \rho^* \vec{u}^* = 0$$ mass

(5) $$\frac{D\rho^*}{Dt} = 0 \implies \nabla \cdot \vec{u}^* = 0$$ continuity

Solution for hydrostatic balance $\vec{u}_o = 0$

$$0 = \frac{\partial p_o}{\partial z} - g\,\rho_o(z)$$

And each dynamic variable is separated

$$\vec{u}^* = \vec{u}_o + \vec{u}$$
$$p^* = p_o + p$$
$$\rho^* = \rho_o + \rho$$

small perturbation from hydrostatic

Thus

(1) (2) (3) $$\frac{\partial \vec{u}}{\partial t} + f\vec{k} \times \vec{u} = -\frac{\nabla \cdot p}{\rho_o} - \frac{g\rho\vec{k}}{\rho_o}$$

(4) $$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho_o}{\partial z} = 0$$

(5) $$\nabla \cdot \vec{u} = 0$$

Try solution $e^{-i\sigma t}$

(1) $-i\sigma u - fv = -p_x / \boxed{\rho_0}$ x-mom

(2) $-i\sigma v + fu = -p_y / \boxed{\rho_0}$ y-mom

(3) • $-i\sigma w = -p_z/\rho_0 - g\rho/\rho_0$ z-mom

(4) $u_x + v_y + w_z = 0$ continuity

(5) • $-i\sigma\rho + w\rho_{0z} = 0$ density, mass

$\rho = \dfrac{w}{i\sigma}\rho_{0z}$ into (3):

(3) $-i\sigma w = -p_z/\rho_0 - \dfrac{g}{\rho_0}\cdot\dfrac{w}{i\sigma}\rho_{0z}$ $\Big|\cdot i\sigma\rho_0$

$$+\sigma^2 w\rho_0 = -\overset{i\sigma}{\vee} p_z \underbrace{-\,g\,w\,\rho_{0z}}_{+\rho_0 N^2 w}$$

(3) or $+p_z i\sigma = \rho_0(N^2 - \sigma^2)\,w$

and horizontal momentum

(1) $$u = \frac{1}{\boxed{\rho_0}} \frac{-i\sigma p_x - f p_y}{\sigma^2 - f^2}$$

(2) $$v = \frac{1}{\boxed{\rho_0}} \frac{-i\sigma p_y + f p_x}{\sigma^2 - f^2}$$

into continuity (4) gives

(4) $$-i\sigma \nabla_H^2 p + (\sigma^2 - f^2) \boxed{\rho_0} w_z = 0$$

$$\nabla_H^2 \cdot = \frac{\partial}{\partial x^2} \cdot + \frac{\partial}{\partial y^2} \cdot$$

$$\frac{\partial}{\partial z} \cdot$$

$$-i\sigma \frac{\partial}{\partial z} \nabla_H^2 p + (\sigma^2 - f^2) \frac{\partial}{\partial z} (\boxed{\rho_0} w_z) = 0$$

$$-i\sigma \nabla_H^2 \underbrace{\frac{\partial}{\partial z} p}_{\rho_0 \frac{(N^2 - \sigma^2) w}{i\sigma} \text{ from (3)}} + (\sigma^2 - f^2)(\boxed{\rho_0} w_z)_z = 0$$

$$\nabla_H^2 \left[\rho_0 (N^2 - \sigma^2) w\right] + (\sigma^2 - f^2)(\rho_0 w_z)_z = 0$$

$$\rho_0 (N^2 - \sigma^2) \nabla_H^2 w + (\sigma^2 - f^2)(\rho_0 w_z)_z = 0$$