

x-mom

$$u = \frac{1}{\rho_0} (\dots)$$

y-mom

$$v = \frac{1}{\rho_0} (\dots)$$

continuity

$$u_x + v_y + w_z = 0$$

$$\left| \frac{\partial}{\partial z} \right)$$

$$-i\sigma \nabla_H^2 \left( \frac{\partial p}{\partial z} \right) + (\sigma^2 - f^2) \left( \frac{p_0}{\rho_0} w_z \right)_z = 0$$

from z-mom

$$-i\sigma w = -\frac{p_z}{\rho_0} - \frac{g}{\rho_0} \cdot \frac{w}{i\sigma} \rho_{0z}$$

$$\left| . \quad \sigma \cancel{w} \right.$$

$$+ \sigma^2 w \cancel{\rho_0} = -i\sigma \frac{p_z}{\rho_0} - \frac{g}{\rho_0} w \rho_{0z}$$

$$N^2 = -\frac{g}{\rho_0} \frac{\partial \rho_0}{\partial z}$$

$$+ \sigma^2 w \cancel{\rho_0} = -i\sigma \frac{p_z}{\rho_0} + \cancel{\rho_0} N^2 w$$

~~$$\cancel{\sigma} + i\sigma p_z = (\sigma^2 - N^2) \cancel{\rho_0 w}$$~~

or

$$i\sigma p_z = (N^2 - \sigma^2) \rho_0 w$$

into  $\frac{\partial}{\partial z}$  (continuity):

$$- \nabla_H^2 \left[ (N^2 - \sigma^2) \rho_0 w \right] + (\sigma^2 - f^2) \left( w_z \frac{p_0}{\rho_0} \right)_z = 0$$

Differential equation in  $w$  only:

~~$$\frac{1}{\rho_0} \left( \frac{p_0}{\rho_0} w_z \right)_z - \frac{(N^2 - \sigma^2)}{(\sigma^2 - f^2)} \nabla_H^2 w = 0$$~~

$$\frac{1}{\rho_0} (\rho_0 \omega_z)_z - \left( \frac{N^2 - \sigma^2}{\sigma^2 - f^2} \right) \nabla_H^2 w = 0$$

$$\frac{\partial^2 w}{\partial z^2} + \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \frac{\partial w}{\partial z} \quad \text{Scaling} \quad \frac{\Delta w}{\Delta z^2}, \quad \frac{1}{\rho_0} \frac{\Delta \rho_0}{\Delta z} \frac{\Delta w}{\Delta z}$$

$$1, \Delta \rho_0 / \rho_0 = 10^{-3}$$

$$\frac{1}{\rho_0} \frac{\partial^2 w}{\partial z^2} \Rightarrow \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \frac{\partial w}{\partial z}$$

### Boussinesq Approximation

"vertical scale of variations for  $w$  is small compared to with the vertical scales for variations of  $\rho_0$ .

$$\boxed{\omega_{zz} - \left( \frac{N^2 - \sigma^2}{\sigma^2 - f^2} \right) \nabla_H^2 w = 0}$$

$\downarrow$   
go to p. 53

$$(6) \text{ At free surface } (z \approx 0) \quad \frac{Dp^*}{Dt} = \sigma \quad p^* = p_0 + p$$

linearize and apply BC at  $z=0$

$$\frac{\partial p^*}{\partial z} + u \frac{\partial p_0}{\partial x} + v \frac{\partial p_0}{\partial y} + w \frac{\partial p_0}{\partial z} = \sigma$$

$$-i\sigma p + w p_{0z} = 0 \quad @ z=0$$

$$-i\sigma p - w g p_0 = \sigma$$

$p_0$  is hydrostatic  
 $p$  is perturbation  
pressure (wave)

$$-i\sigma \nabla_H^2 p - g p_0 \nabla_H^2 w = 0$$

$$\frac{\partial p_0}{\partial z} = -p_0 g$$

1

$\nabla_H^2$  1

Recall that from continuity (4) on p. 50 we had

$$-i\sigma \cdot \nabla_H^2 p = -(\sigma^2 - f^2) \rho_0 w_z$$

Then the BC @  $z=0$  becomes

$$-\infty - (\sigma^2 - f^2) \rho_0 w_z - g \rho_0 \nabla_H^2 w = 0 \quad @ z=0$$

or

$$(6) \quad (\sigma^2 - f^2) w_z + g \nabla_H^2 w = 0 \quad @ z=0$$

$$(7) \quad \text{And BC @ flat bottom } w = 0 \quad @ z=-D$$

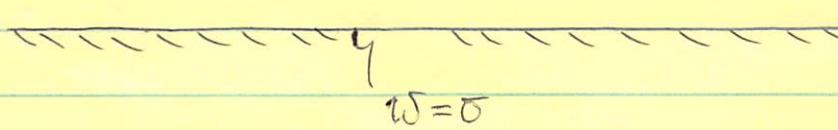
So, our wave problem is

$$(\sigma^2 - f^2) w_z + g \nabla_H^2 w = 0$$



$$z = -D$$

$$w_{zz} - \left( \frac{N^2 - \sigma^2}{\sigma^2 - f^2} \right) \nabla_H^2 w = 0 \quad \text{with } N^2 = N^2(z)$$



$$z = 0 \quad z = -D$$

Unbounded, rotating, stratified fluid

Assume  $N^2(z) = -\frac{g}{\rho_0} \frac{\partial \rho_0}{\partial z} = \text{const}$  linear vertical density gradient

and  $N^2 > f^2$  with  $f = \text{const.}$  also

Then the interior governing equation has only constant coefficients

$$\omega_{zz} - \frac{(N^2 - \sigma^2)}{\sigma^2 - f^2} (\omega_{xx} + \omega_{yy}) = \sigma$$

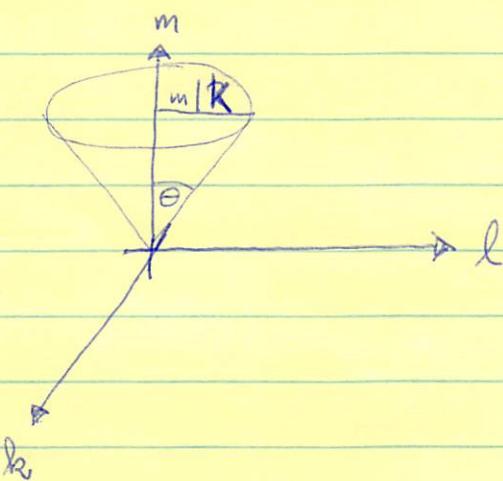
with solutions

$$\omega = e^{-i\sigma t + ikx + iky + imz}$$

to get dispersion

$$m^2 - \frac{(N^2 - \sigma^2)(k^2 + l^2)}{(\sigma^2 - f^2)} = \sigma$$

For  $N^2 > \sigma^2 > f^2$  this dispersion describes a cone



where all possible wave vectors with frequency  $\sigma$  lie on the surface of this cone.

$$K = \sqrt{k^2 + l^2 + m^2}$$

## Dispersion

$$m^2 = \frac{N^2 - \sigma^2}{\sigma^2 - f^2} (k^2 + l^2)$$

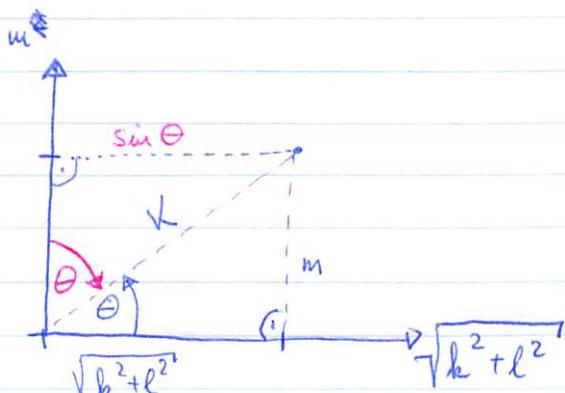
$$m^2 \sigma^2 - m^2 f^2 = N^2 (k^2 + l^2) - \sigma^2 (k^2 + l^2)$$

$$\therefore \sigma^2 (m^2 + k^2 + l^2) = N^2 (k^2 + l^2) + f^2 m^2$$

$$1 \boxed{\sigma^2 = \frac{N^2 (k^2 + l^2) + f^2 m^2}{(m^2 + k^2 + l^2)}}$$

$$= N^2 \frac{k^2 + l^2}{m^2 + k^2 + l^2} + f^2 \frac{m^2}{m^2 + k^2 + l^2}$$

$$= N^2 \sin^2 \theta + f^2 \cos^2 \theta$$



$$K^2 = m^2 + k^2 + l^2$$

$$\frac{\sin}{\cos} \theta = \sqrt{\frac{k^2 + l^2}{m^2}} ; \quad \frac{\sin^2}{\cos^2} = \frac{(k^2 + l^2)}{K^2}$$

$$\frac{\cos}{\sin} \theta = \frac{k^2 + l^2}{m^2} ; \quad \frac{\cos^2}{\sin^2} = \frac{m^2}{K^2}$$

$$\theta = \tan^{-1} \frac{m}{\sqrt{k^2 + l^2}}$$

Waves of form  $(\vec{u}, p, \rho) = (\vec{u}_0, p_0, \rho_0) e^{-i\omega t + i\vec{k} \cdot \vec{x}}$

A

(1) From continuity  $\nabla \cdot \vec{u} = 0$  we get  $\vec{k} \cdot \vec{u}_0 = 0$

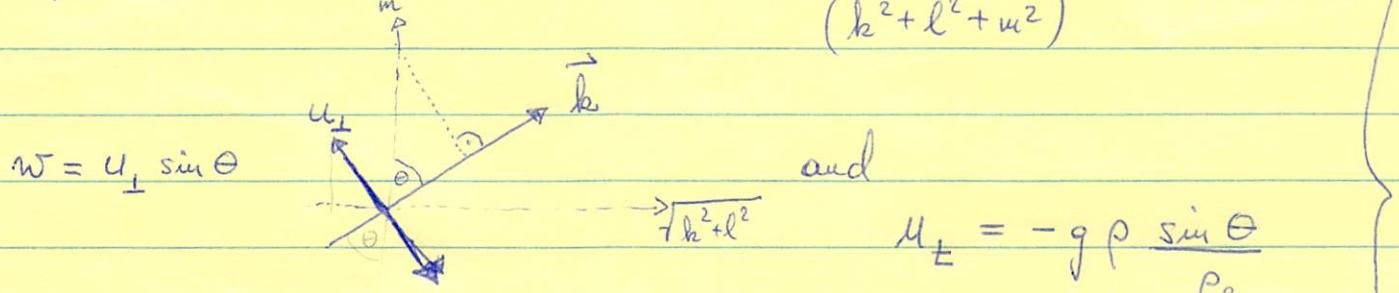
that tells us fluid motion  $\vec{u}_0 \perp$  wave propagation  $\vec{k}$

[new: In sound waves and surface gravity waves we had  $\vec{k} \parallel \vec{u}$ ]

(2)  $\nabla p \sim \vec{k} \cdot \vec{p}_0$

that tells us the  $p$ -gradient  $\perp$  fluid motions

$$(3) f = 0 \quad \text{then} \quad \nabla^2 = N^2 \frac{(k^2 + l^2)}{(k^2 + l^2 + m^2)} = N^2 \sin^2 \theta$$



$$w = u_\perp \sin \theta$$

and

$$M_\perp = -g \rho \frac{\sin \theta}{\rho_0}$$

rotate into  $\vec{k}$ , that gives

$$\vec{u} = (u_\parallel, u_\perp) = (0, u_\perp)$$

$$\rho_z + \mu \sin \theta \frac{\partial p}{\partial z} = 0$$

since pressure gradient is in direction  $\parallel$  to  $\vec{k}$ , the motion  $u_\perp$  is decoupled from pressure gradient

Phase Velocity

$$\vec{c}_p = \frac{\omega}{\vec{k}} = \frac{\omega \cdot \vec{k}}{|\vec{k}|^2}$$

$$\omega^2 = \frac{N^2 (k^2 + l^2) + f^2 m^2}{m^2 + k^2 + l^2}$$

$$f = \omega \quad \omega = \frac{N}{|\vec{k}|} \sqrt{k^2 + l^2}$$

↑

$$\vec{c}_p = \frac{N}{|\vec{k}|^3} (k^2 + l^2)^{1/2} \vec{k}$$

Group Velocity

$$\vec{c}_g = (c_{gx}, c_{gy}, c_{gz})$$

$$c_{gx} = \frac{\partial \omega}{\partial k_x} = \frac{l}{\omega} \frac{(N^2 - \omega^2)}{|\vec{k}|^2}$$

$$c_{gy} = \frac{\partial \omega}{\partial k_y} = \frac{l}{\omega} \frac{(N^2 - \omega^2)}{|\vec{k}|^2}$$

$$c_{gz} = \frac{\partial \omega}{\partial m} = \frac{-m}{\omega} \frac{\omega^2 - f^2}{|\vec{k}|^2} = \frac{-m}{\omega} \cdot \frac{\omega^2}{|\vec{k}|^2} = -\frac{m \omega}{|\vec{k}|^2}$$

↑  
 $f = \omega$

Notice

$$\vec{k} \cdot \vec{c}_g = \left( \frac{k^2}{\sigma} + \frac{l^2}{\sigma} \right) \left( \frac{N^2 - \sigma^2}{K^2} \right) - \frac{m^2}{\sigma} \left( \frac{\sigma^2 - f^2}{K^2} \right)$$

$$= 0$$

$$K^2 = |\vec{k}|^2 = (k^2 + l^2 + m^2)$$

$$A \quad \vec{c}_g \perp \vec{k}$$

$$\vec{c}_p \parallel \vec{k}$$

$$A \quad \vec{c}_g \perp \vec{c}_p$$

Another way to write the group velocity

$$\vec{c}_g = \frac{1}{\sigma K^2} \begin{pmatrix} k(N^2 - \sigma^2) \\ l(N^2 - \sigma^2) \\ m(f^2 - \sigma^2 + N^2 - N^2) \end{pmatrix} = \cancel{\frac{1}{\sigma K^2}}$$

$$= \frac{1}{\sigma K^2} \left[ (N^2 - \sigma^2) \begin{pmatrix} k \\ l \\ m \end{pmatrix} \cancel{+} (N^2 - f^2) \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix} \right]$$

Perhaps this helps to visualize  $\vec{c}_g$