## 1. "Coriolis Force:" Mathematical Derivation

The Coriolis force emerges from Newton's 2nd Law valid in a non-rotating (inertial) reference frame

$$
\begin{equation*}
\vec{F}=\operatorname{mass} \cdot\left(\frac{D^{2}}{D t^{2}} \vec{r}\right)_{i n} \tag{1}
\end{equation*}
$$

when velocities and accelerations are measured from within a reference frame that is rotating at a constant rate $\vec{\Omega}$. Here $\vec{r}$ is the position vector of a particle that is the same in either inertial or rotating system. The change over time of any vector $\vec{B}$, however, has different descriptions when measured in a non-rotating ("in") or rotating ("rot") reference frame:

$$
\begin{equation*}
\left(\frac{D}{D t} \vec{B}\right)_{\text {in }}=\left(\frac{D}{D t} \vec{B}\right)_{r o t}+\vec{\Omega} \times \vec{B} \tag{2}
\end{equation*}
$$

The velocity $\vec{u}$ is the first derivative of position $\vec{r}$ which gives

$$
\begin{equation*}
\left(\frac{D}{D t} \vec{r}\right)_{i n}=\left(\frac{D}{D t} \vec{r}\right)_{r o t}+\vec{\Omega} \times \vec{r} \tag{3}
\end{equation*}
$$

or

$$
\begin{equation*}
\vec{u}_{i n}=\vec{u}_{r o t}+\vec{\Omega} \times \vec{r} \tag{4}
\end{equation*}
$$

The acceleration $\vec{a}$ is the first derivative of velocity $\vec{u}$ or second derivative of $\vec{r}$ which gives

$$
\begin{equation*}
\left(\frac{D}{D t} \vec{u}_{i n}\right)_{i n}=\left(\frac{D}{D t} \vec{u}_{i n}\right)_{r o t}+\vec{\Omega} \times \vec{u}_{i n} \tag{5}
\end{equation*}
$$

Inserting (4) into (5) gives

$$
\begin{gather*}
\left(\frac{D}{D t} \vec{u}_{i n}\right)_{i n}=\left(\frac{D}{D t}\left(\vec{u}_{r o t}+\vec{\Omega} \times \vec{r}\right)\right)_{r o t}+\vec{\Omega} \times\left(\vec{u}_{r o t}+\vec{\Omega} \times \vec{r}\right)  \tag{6}\\
\left(\frac{D}{D t} \vec{u}\right)_{i n}=\left(\frac{D}{D t} \vec{u}\right)_{r o t}+\vec{\Omega} \times\left(\frac{D}{D t} \vec{r}\right)_{r o t}+\vec{\Omega} \times \vec{u}_{r o t}+\vec{\Omega} \times \vec{\Omega} \times \vec{r}  \tag{7}\\
\left(\frac{D}{D t} \vec{u}\right)_{i n}=\left(\frac{D}{D t} \vec{u}\right)_{r o t}+\vec{\Omega} \times \vec{u}_{r o t}+\vec{\Omega} \times \vec{u}_{r o t}+\vec{\Omega} \times \vec{\Omega} \times \vec{r} \tag{8}
\end{gather*}
$$

We will discuss the terms on the right next as the Coriolis $\left(2 \vec{\Omega} \times \vec{u}_{r o t}\right)$ and Centrifugal accelerations $(\vec{\Omega} \times \vec{\Omega} \times \vec{r})$, respectively.

## 2. Vector Cross-Products, Gradients, and Potentials

We write the centrifugal acceleration as the double vector product

$$
\vec{\Omega} \times(\vec{\Omega} \times \vec{r})=\left(\begin{array}{l}
0  \tag{9}\\
0 \\
\Omega
\end{array}\right) \times\left[\left(\begin{array}{l}
0 \\
0 \\
\Omega
\end{array}\right) \times\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)\right]
$$

evaluating the first cross-product gives

$$
\vec{\Omega} \times(\vec{\Omega} \times \vec{r})=\left(\begin{array}{l}
0  \tag{10}\\
0 \\
\Omega
\end{array}\right) \times\left(\begin{array}{c}
-\Omega \cdot y \\
+\Omega \cdot x \\
0
\end{array}\right)
$$

and then

$$
\vec{\Omega} \times(\vec{\Omega} \times \vec{r})=\left(\begin{array}{c}
-\Omega^{2} \cdot x  \tag{11}\\
-\Omega^{2} \cdot y \\
0
\end{array}\right)=-\Omega^{2} \cdot\left(\begin{array}{l}
x \\
y \\
0
\end{array}\right)
$$

which gives several identities

$$
\vec{\Omega} \times(\vec{\Omega} \times \vec{r})=-\Omega^{2} / 2 \cdot \nabla\left(\vec{r}_{\perp} \cdot \vec{r}_{\perp}\right)=-\Omega^{2} / 2 \cdot\left(\begin{array}{c}
\frac{\partial}{\partial x}  \tag{12}\\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{array}\right)\left(x^{2}+y^{2}\right)=-\Omega^{2} / 2 \cdot\left(\begin{array}{c}
2 x \\
2 y \\
0
\end{array}\right)
$$

and, finally,

$$
\begin{equation*}
\vec{\Omega} \times(\vec{\Omega} \times \vec{r})=-\nabla \cdot \Theta_{c} \tag{13}
\end{equation*}
$$

We write the centrifugal acceleration as a gradient $\nabla \cdot$ of the scalar potential $\Theta_{c}=\left(\Omega \cdot \vec{r}_{\perp}\right)^{2} / 2$ where $\vec{r}_{\perp}=(x, y, 0), x=r \cdot \cos (\phi)$ while r and $\phi$ are the earth's radius and latitude.

We can write gravity, too, as the gradient of a scalar potential $\Theta_{g}=g \cdot|\vec{r}|$ and the gravitational acceleration is

$$
\begin{equation*}
\vec{g}=-\nabla \cdot \Theta_{g} \tag{14}
\end{equation*}
$$

where the length (or magnitude of $\vec{r}$ is $|\vec{r}|=\sqrt{\left(x^{2}+y^{2}+z^{2}\right) \text { to give } \nabla \cdot|\vec{r}|=\vec{r} /|\vec{r}| \text {. Combining } \quad \text {. }{ }^{2} \text {. }}$ the two accelerations of Eq.-13 and Eq.-14, we define an effective gravity that is modified by
the centrifugal acceleration of the rotating reference system

$$
\begin{equation*}
\vec{g}_{e f f}=-\nabla \cdot\left(\Theta_{c}+\Theta_{g}\right) \tag{15}
\end{equation*}
$$

that is sketched in Figure-1 in the ( $\mathrm{x}, \mathrm{z}$ ) plane. The "effective weight" vector indicates the effective gravity. And, yes, you loose weight moving towards the equator where the centrifugal acceleration has its maximum while you gain weight moving towards the poles where the centrifugal acceleration is zero.


FIG. 1. Vectors of centrifugal, gravity, and effective gravity in ( $\mathrm{x}, \mathrm{z}$ ) plane of a rotating earth.

