Spin-up of a stratified ocean, with applications to upwelling

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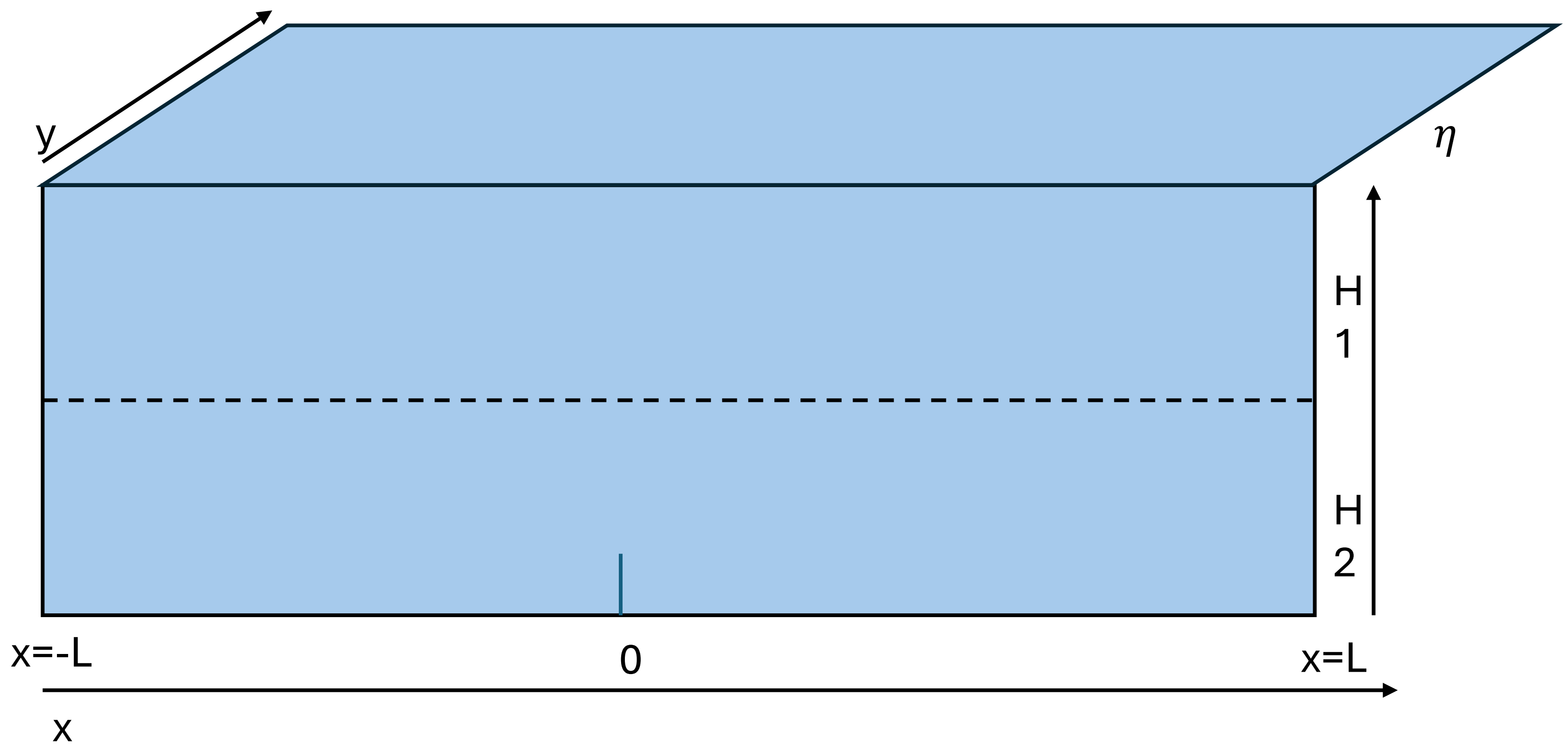
Overview

- How does an ocean, initially stratified but at rest respond when a steady wind stress is suddenly applied?
- Do the barotropic and baroclinic modes respond differently?
- Also explore the behavior of upwelling in a North-South applied wind-stress scenario.

- Previous work (Pedlosky, Phillips, Veronis, Lighthill) have studied various aspects of boundary effects and stratification.
- This paper presents a unified treatment summing the ocean spin-up case in terms of planetary waves.


Putting the Motion in the Ocean

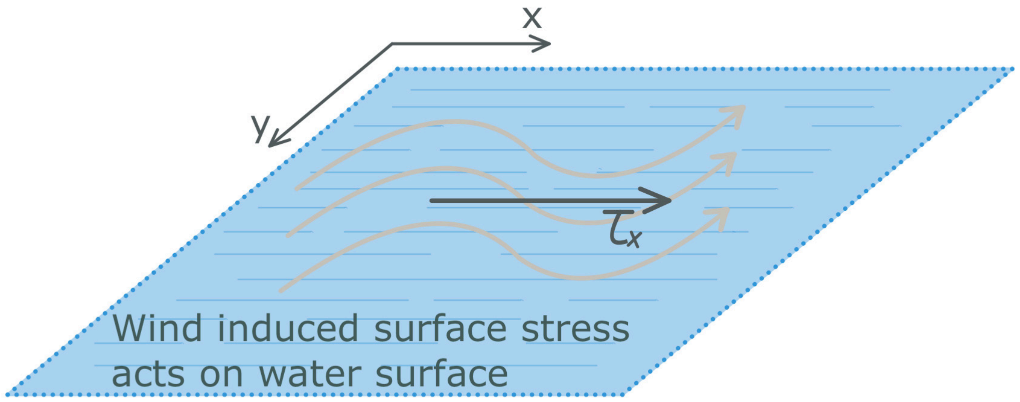
Initial Model Setup



- Each case, we start with an idealized ocean at rest
- Ocean contained between 2 meridional boundaries:
  - x=-L
  - x=L
- Wind stress with periodic behavior in y assumed
- Two layers, H1 and H2
- Yields two modes describing the vertical structure, barotropic and baroclinic

Approximation

- Assumption: adjustments take place on time scales large compared to inertial time scale: $\frac{1}{2\Omega \sin \theta}$
- Assumption: wind stress τ^x *and* τ^y are of the same order and each can be treated independently
- Boundary layers (such as upwelling), so not geostrophic balance
  - flows parallel to coast balanced by pressure gradient (quasi-geostrohpic)
  - flows normal to coast – quasi-geostrophic approximation doesn't work – a little Gill magic to get approximation that works



Model Parameters

Barotropic

$$u = \frac{H_1 u_1 + H_2 u_2}{H_1 + H_2}$$

$$p = g\eta$$

$$H = H_1 + H_2$$

$$c^2 = gH$$

c ~ 200 m/s

Baroclinic

$$u = u_1 - u_2$$

$$H = \frac{H_1 H_2}{H_1 + H_2}$$

$$p = \acute{g}h$$

$$\acute{g} = (\rho_2 - \rho_1)\frac{g}{\rho_2}$$

$$c^2 = \acute{g}H$$

c ~ 2 m/s

Beta-plane Approximation

$$fu = -p_y - \frac{\beta}{f} p - \frac{1}{f} p_{xt} + \frac{\tau^y}{H}$$

$$fv = -p_x - \frac{1}{f} p_{yt} + \frac{\tau^x}{H}$$

$$p_t + c^2 (u_x + v_y) = 0$$

Recall the Coriolis and beta-plane parameters

$$f = 2\Omega \sin \theta_0$$

$$\beta = 2\Omega \sin \frac{\theta_0}{R}$$

- Recall that the beta-plane accounts for variation in latitude. In the approximation, we assume latitude departures from a given latitude are small.
- Anderson and Gill step through the derivation of the approximation equations starting from spherical coordinates to the equations to the left.
- We don't need to revisit that math this close to the end of the semester.

General Solutions

$$(u_{xx} - \mu^2)_t + \beta u_x = -\frac{l^2}{H}(1 - i\varepsilon)\tau^x$$

where:

$$\mu^2 = \left(\frac{f}{c}\right)^2 + l^2$$

and:

$$\varepsilon = \frac{\beta}{fk}$$

and solutions depend on:

$$\Lambda = \mu^2 L^2$$

- Barotropic mode: $1/_{\mu} \sim 1/_{l} \sim 1000\ km$
  - L ~ 3000 km, Λ~10
- Baroclinic mode: $1/_{\mu} \sim c/_{f} \sim 30\ km$
  - baroclinic radius of deformation*
  - Λ~10,000

East-West Wind Stress

- Two types of wave-like solutions (Lighthill, 1969):
  - Rossby waves with frequency < $\beta/2\mu$
  - Kelvin waves with frequency > $\beta/2\mu$

* Good reference on radius of deformation for barotropic and baroclinic waves in the oceans.

https://ceoas.oregonstate.edu/rossby_radius#:~:text=The%20baroclinic%20Rossby%20radius%20of%20deformation%20is%20directly%20related%20to,study%20of%20ocean%20wave%20dynamics.

East-West Scaled Solution

Scaling:

$L \; for \; x$

$\frac{1}{\beta L} \; for \; t$

$\frac{Ll^2(1-i\varepsilon)\tau^x}{\beta H} \; for \; u$

After scaling we get:

$$(u_{xx} - \Lambda u)_t + u_x = 1$$

- Boundary conditions:
  - u = 0 at x = 1 and -1
- Sverdrup's solution (time-independent):
  - $u_x = 1$
- Spate-independent solution (no boundary):
  - $u = -t/\Lambda$

From the planetary wave equation (left) we get:

$$C_g = \frac{k^2 - \Lambda}{(k^2 + \Lambda)^2}$$

- largest group velocity at k=0
- as k increases, Cg changes sign and increases to $\frac{1}{8}\Lambda$ at $k = \sqrt{3\Lambda}$, then zeros out as k tends to infinity

Key Concepts:

- waves from eastern boundary rapidly move westward by long waves
- waves from the western boundary move eastward at 1/8 the speed by short waves
- interior is Sverdrup's solution as the long wave front dominates

Western Boundary Layer

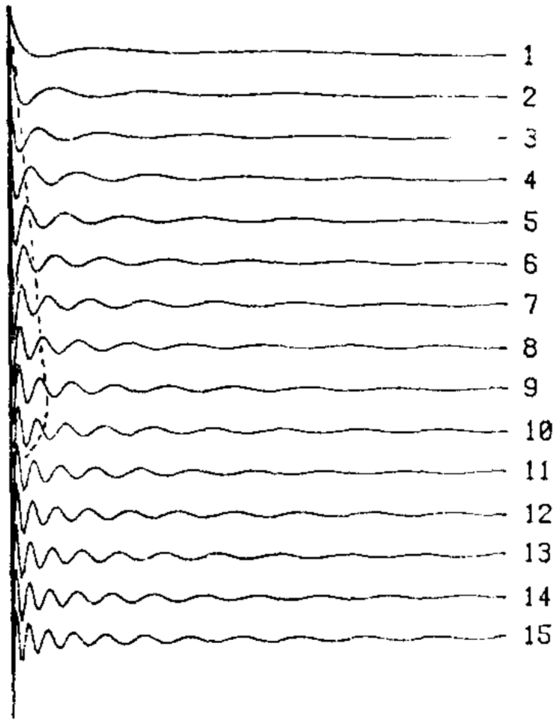


Fig. 1. Plot of (7.3) as a function of x for various values of t. The value of Λ is 600. The region to the left of the dashed line is that in which the above function is a good approximation to the numerically determined solution of (6.4). The plotting frequency is 100 units of dimensionless time.

- Eastward barotropic wave scale ~ 600 km
- Eastward baroclinic wave scale ~ 20 km
- Western boundary solution becomes dominated by even shorter waves as time progresses
  - waves have smaller group velocity
  - remain near western boundary
  - boundary layer gets thinner with time (width inversely proportional to time)
- Approximation where u vanishes at x = -1
- Solution includes a Bessel function

Short-Wave Approximation:

$$u_{xxt} + u_x = 0$$

with solution:

$$u = -\frac{1}{\Lambda}\left\{t - \left(\frac{t}{1+x}\right)^{\frac{1}{2}} J_1\left[2\sqrt{(1+x)t}\right]\right\}$$

Western Boundary Layer

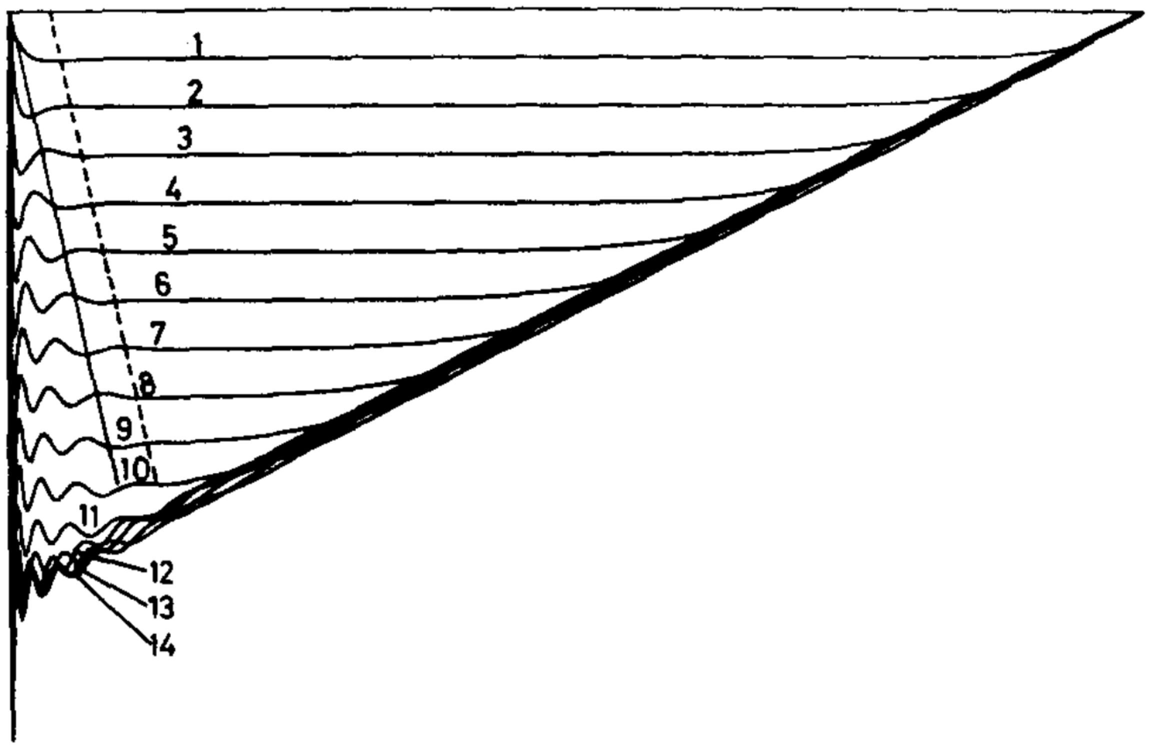
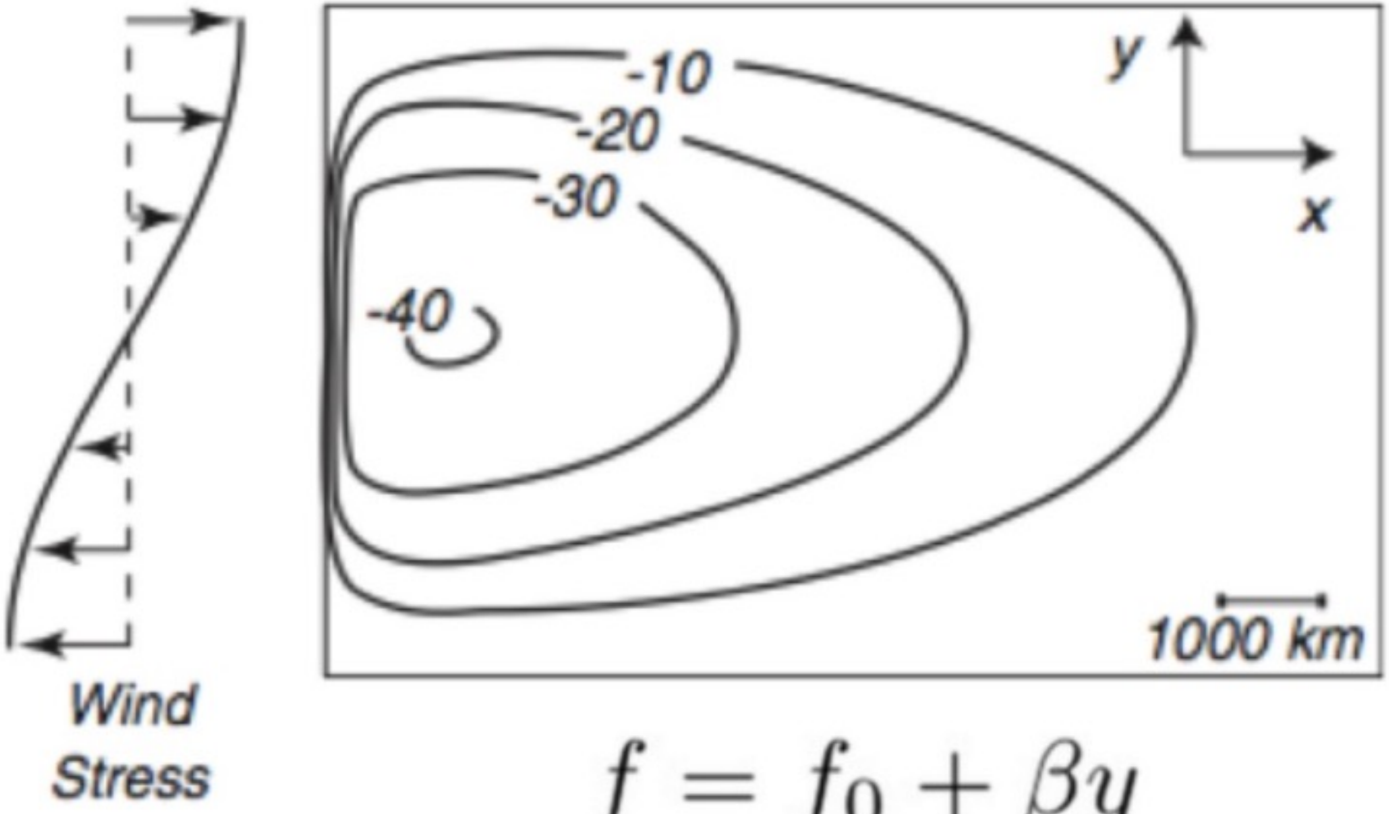


Fig. 2. The numerically determined solution of (6.4) for comparison with Fig. 1. The solid diagonal line denotes the maximum distance information from the western boundary could have been propagated eastward into the interior by planetary waves. The dashed line is displaced a distance δ from the solid line.

Western Boundary Layer



$$f = f_0 + \beta y$$

- Beta-plane approximation – varying Coriolis
- Asymmetric gyre - Sverdrup
- Western boundary current dominated by short waves
- After a time 2Λ the solutions to the boundary case a no longer valid as the Sverdrup solution dominates

Western Boundary Layer

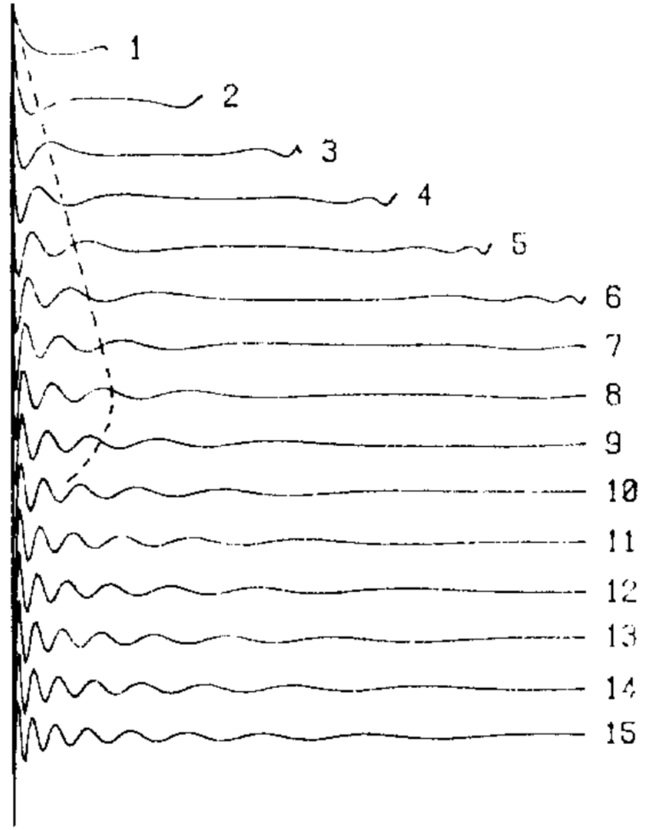


Fig. 3. Plot of (7.6) for comparison with Figs. 1 and 2. The region to the left of the dashed line is that in which the above function is a good approximation to that of Fig. 2.

- Eventually Sverdrup or interior solution dominates and growth of the short waves in the western boundary current cease
- Dotted line – finite distance of which the western boundary current is trapped
- Caveat – results show a system that is incompressible to inertial gravity waves
- Trapping scale (incompressible to inertial gravity waves):
  - $\delta = \Lambda^{-\frac{1}{2}}$ (baroclinic radius of deformation)
- Finite distance affected by western boundary solution:
  - $d_w = \delta + \frac{t}{8\Lambda}$

Western Boundary Layer

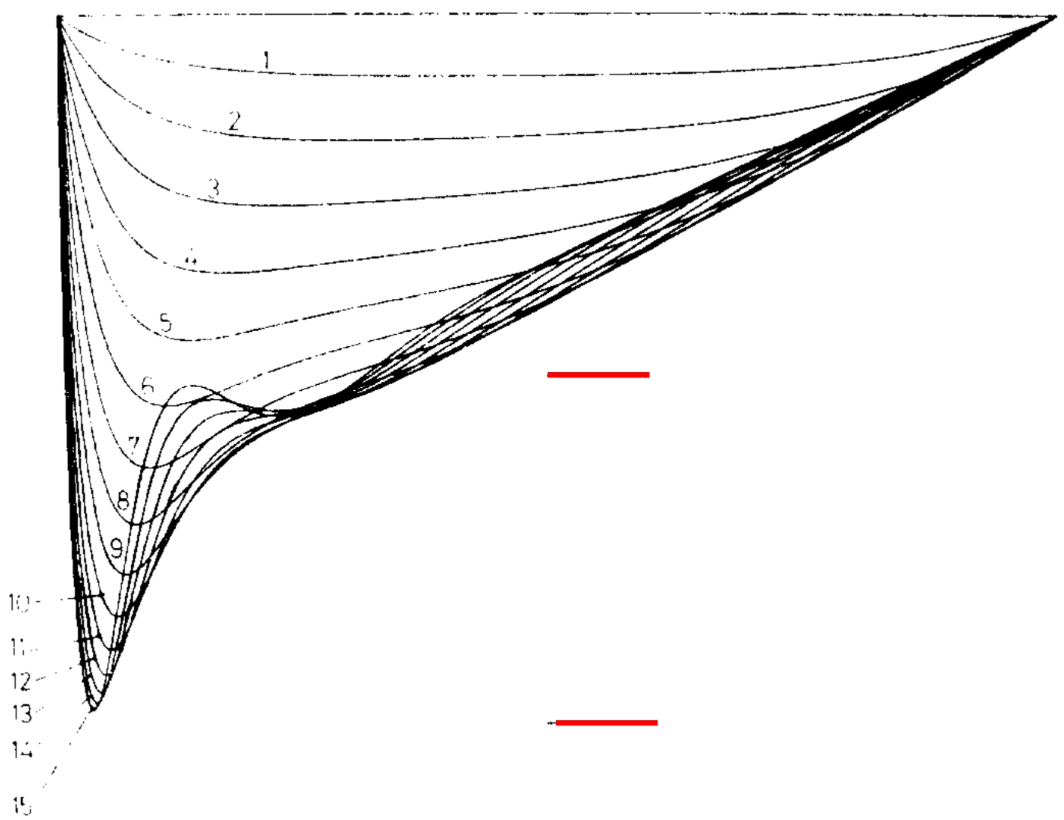


Fig. 4. Numerically determined solution of (6.4) for $\Lambda = 20$. There is no obvious J_1 spin-up solution for this case because the Sverdrup solution is established across the whole ocean before such a boundary layer can become well established. The J_0 solution which occurs for large time is apparent, however. Solid horizontal lines are at $u = -2{\cdot}8$, and $u = -1{\cdot}4$. The plotting frequency is 5 units of dimensionless time.

- Results for $\Lambda = 20$
- Max: u1 = -2.8
- Min: u2 = -1.4
- Figure shows max and min (red lines) and good approximation over time

Western Boundary Layer

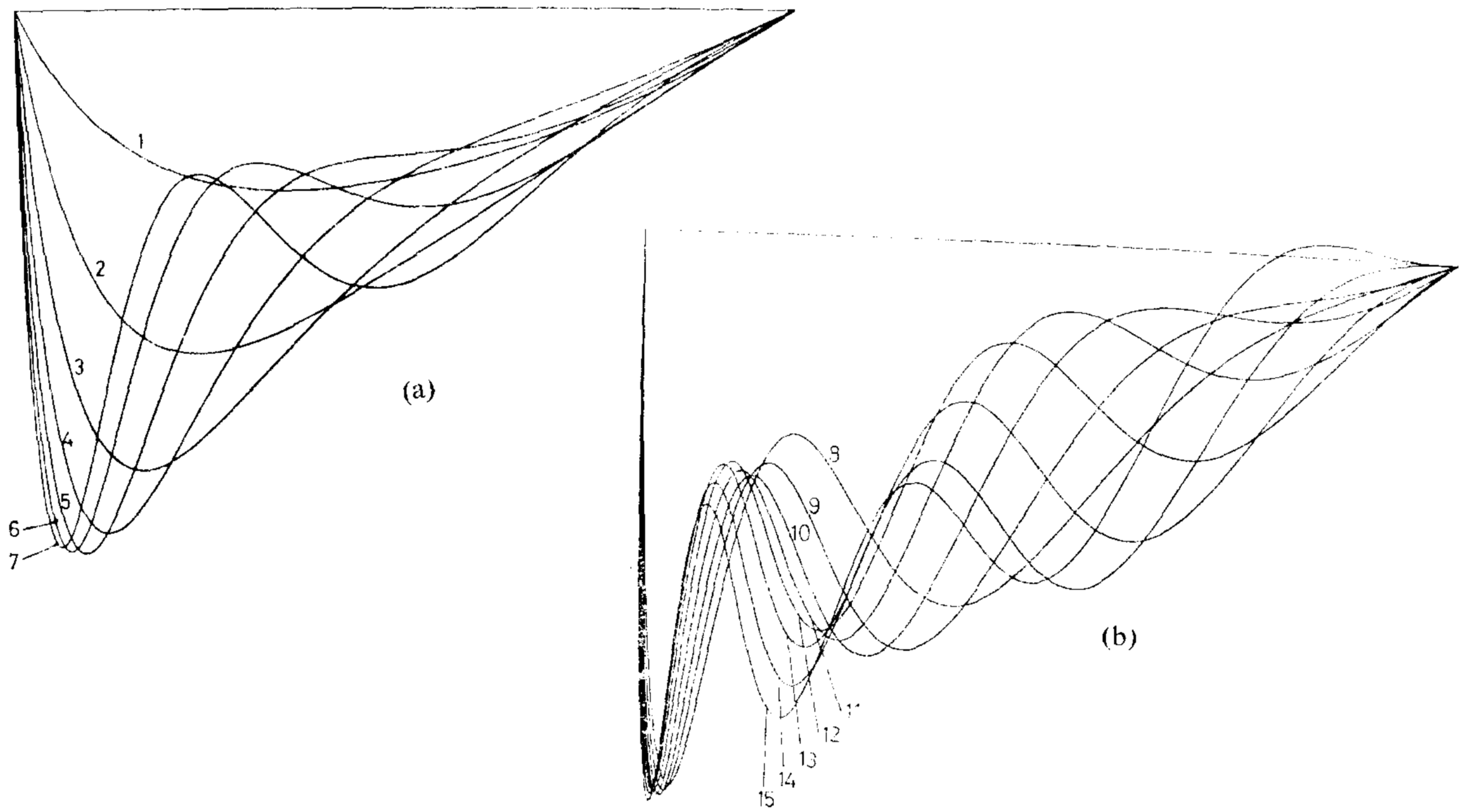


Fig. 5a, b. As for Figs. 2, 4 but for $\Lambda = 4$. Note the larger amplitude perturbations across the whole width of the basin. Curves plotted every 5 units of dimensionless time.

- Results for $\Lambda = 4$ (barotropic wave)
- Large oscillations extend over most of the basin, still decreasing eastward
- For smaller Λ, the oscillation can make u reverse near the eastern boundary meaning a short wave reached the eastern boundary from the west

North-South Wind Stress: f-plane

$$(u_{xx} - \mu^2)_t + \beta u_x = -\frac{l^2}{H}(1 - i\varepsilon)\tau^x$$

becomes:

$$(u_{xx} - \mu^2)_t + \beta u_x = -\frac{\mu^2 \tau^y}{fH}\delta(t)$$

and solutions for f-plane:

$$u = -\frac{\tau^y}{fH}\left(\frac{\cosh \mu x}{\cosh \mu L} - 1\right)$$

where:

$$\mu^2 = \left(\frac{f}{c}\right)^2 + l^2$$

- Consider only impulse effect at t=0, otherwise at t > 0, right hand side of top equation vanishes
- In the f-plane solution for the baroclinic mode (μL is large):
  - u is constant (Ekman flux)
  - at boundaries, narrow layer where u goes to 0 – trapping distance (1/u)
  - this creates divergence and movement of the thermocline, results in upwelling and costal jet (long shore) from thermocline slope (geostrophic)

North-South Wind Stress: beta-plane

Choosing scales:

$L \; for \; x$

$\frac{1}{\beta L} \; for \; t$

$\frac{\tau^y}{fH} \; for \; u$

Yields this equation for u, and for t > 0:

$$(u_{xx} - \Lambda u)_t + u_x = 0$$

and solutions for beta-plane:

$$u = 1 - \frac{\cosh \sqrt{\Lambda}\, x}{\cosh \sqrt{\Lambda}}$$

- Initial condition – impulsively generated at t=0
- In the beta-plane setup for the baroclinic mode:
  - over longer term solution – planetary waves allow energy to leak westward
  - coastal boundary layer no longer confined to f-plane trapping distance (Rossby deformation)

North-South Wind Stress

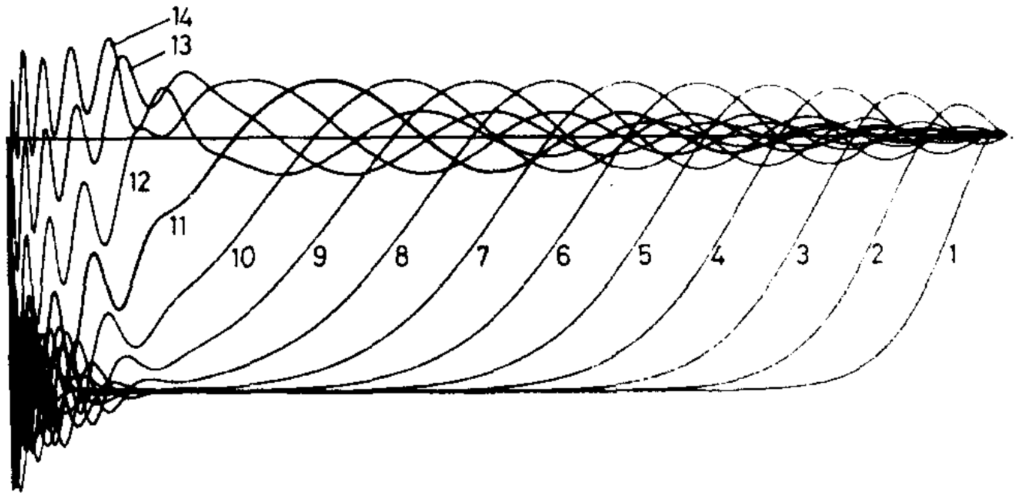


Fig. 7. Relaxation of the *f*-plane solution (9.3) for a steady longshore wind stress independent of x, y when β effects are included (9.2). The propagation of Rossby waves from the east is clearly visible. The solution relaxes to the Sverdrup solution $u = 0$. Curves are plotted every 100 units of dimensionless time.

- Initial planetary (long) waves propagate from east to west quickly
- They leave a wake behind the primary front
- Eventually settle to the Sverdrup solution
- Time period (real time) for the system to relax is 3 months ($\frac{f}{\beta} \sim 5000\ km, c \sim 2.5\ m\ s^{-1}$)
  - cyclone-scale upwelling unaffected by beta-effect (Kelvin/f-plane best approach)
  - seasonal upwelling is on this scale

Thermocline Displacement

- Move from solutions for u (eastward velocity component) to p (pressure)
- Again, two-layer model, with a deep lower layer
- Thermocline displacement given by:
  - $h \sim p/\acute{g}\rho$
- Examine f-plane and beta-plane

Thermocline Displacement: f-plane

Solution to our equations of motion valid at the coastal boundary layers:

$$p = \frac{\tau^y}{lH} e^{\mu(x-L)} \left[\sin ly - \sin l \left(y - \frac{ft}{\mu} \right) \right]$$

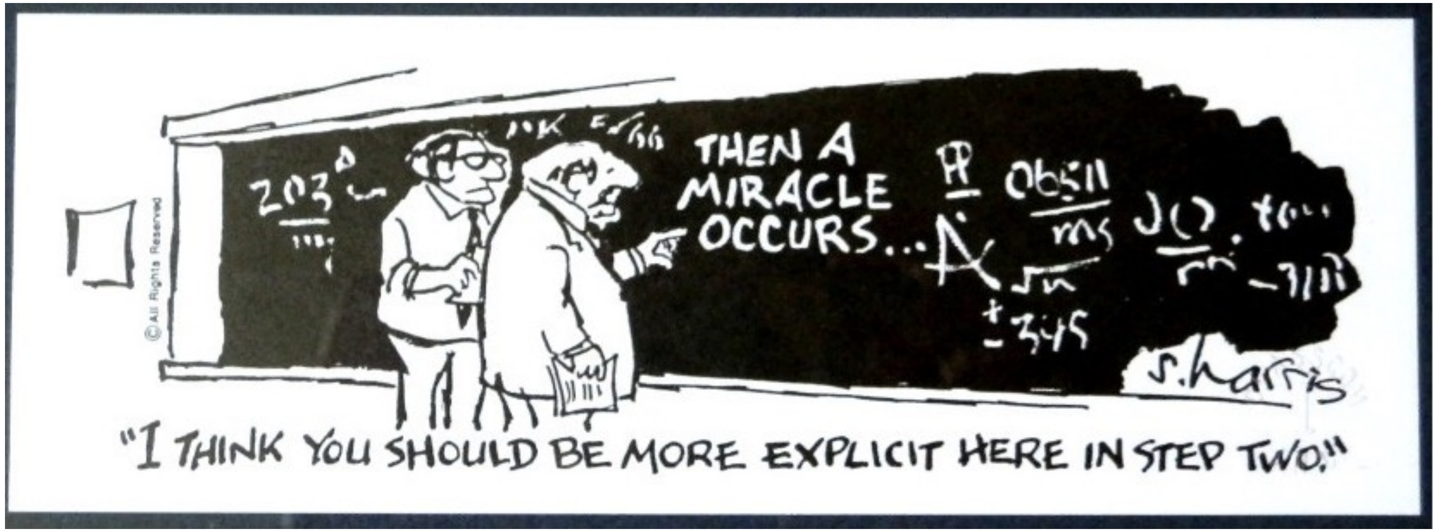
- This is a northward traveling Kelvin wave (Northern Hemisphere)
- Initially: two sinusoids cancel and p=0
- p increases linearly with time and two sinusoids get out of phase
- Leads to a max value of p:
  - $p_{max} = \frac{2\tau^y}{lH}$
- Successive waves pass along the coast at intervals of:
  - $\frac{\pi\mu}{fl} = \frac{\pi}{cl}$
- Maximum upwelling is a quarter wavelength poleward of the maximum equatorward wind
  - time-average p at max when $y = \frac{2\pi}{l}$
  - wind is max at $y = 0$

Thermocline Displacement: beta-plane

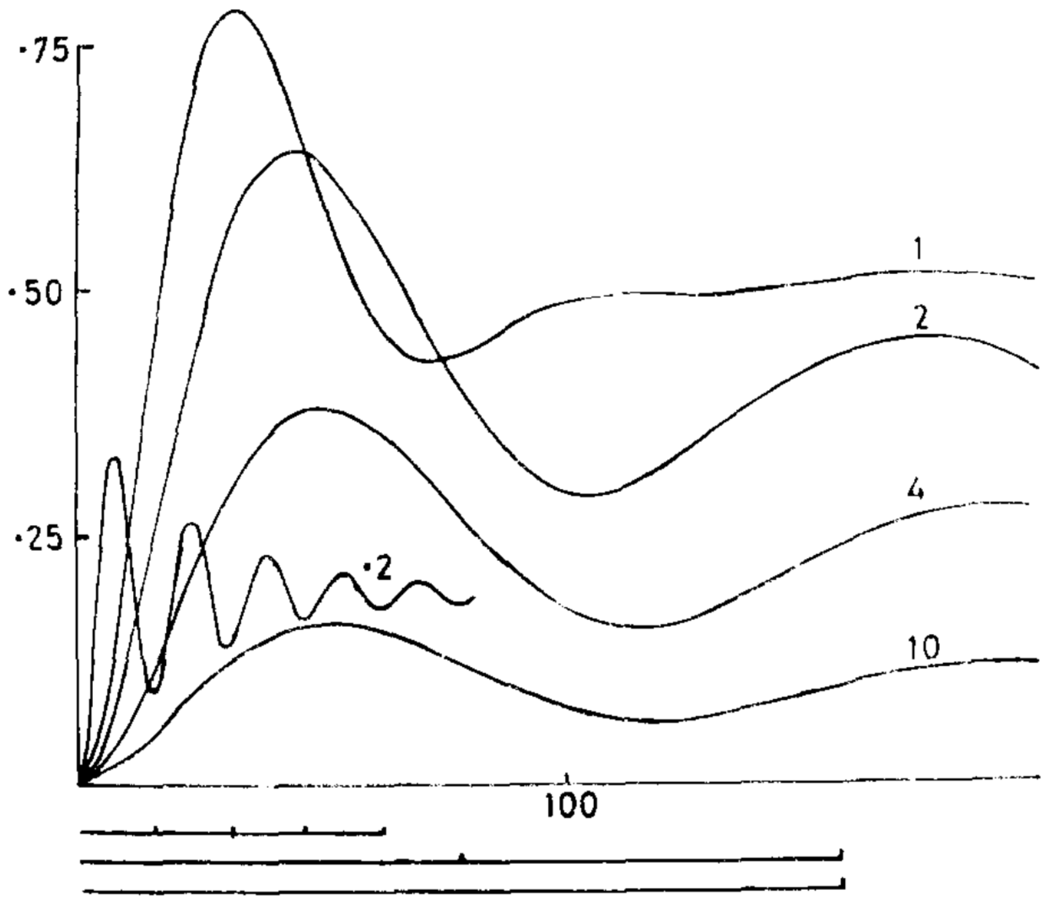
Solution to our equations of motion valid at the coastal boundary layers:

$$\frac{\beta H p}{f\tau} = \frac{1}{1 + i/\varepsilon} - \frac{2i}{\pi}\int_{1}^{\infty} \frac{\cos\left(\frac{\beta c t}{2f}R\right)}{R}\left[\frac{1}{\left(1 + \frac{2i}{\varepsilon} - i\sqrt{R^2 - 1}\right)} - \frac{1}{\left(1 + \frac{2i}{\varepsilon} + i\sqrt{R^2 - 1}\right)}\right] dR$$

- Baroclinic mode:
  - μL is large
  - $\mu \sim \frac{f}{c}$
- Yields two time-scales
  - $\frac{1}{cl}$ - time scale of Kelvin waves
  - $\frac{f}{\beta c}$ - time scale of planetary waves
- Ratio of time scales
  - $\varepsilon = \frac{\beta}{fl}$, which is normally small
  - planetary waves effects take place longer



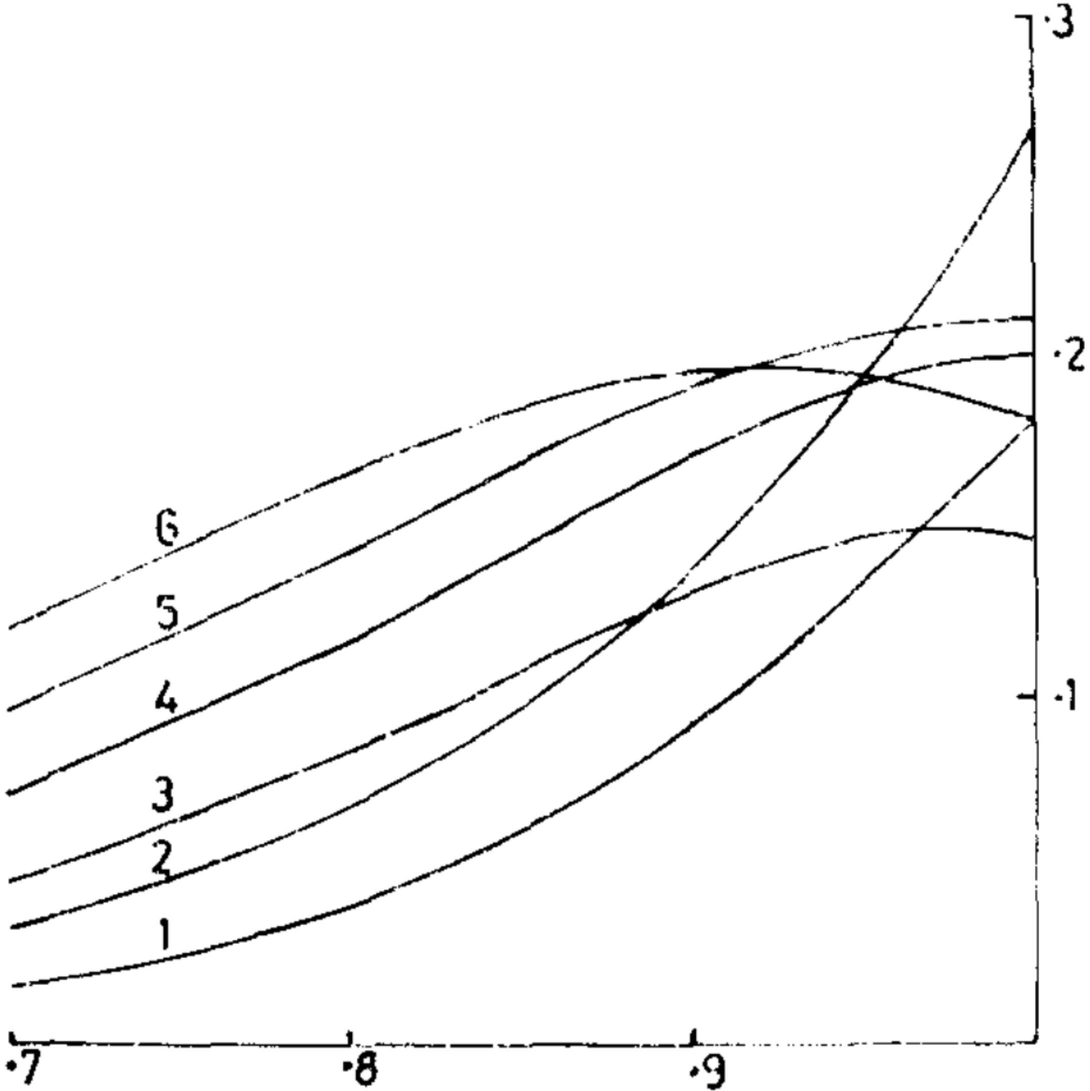
Thermocline Displacement: beta-plane



- Even for small times, the damped oscillation suggests that the 3-month time scale for planetary wave effects to upwelling is an overestimate.
- For longer times, integral gets small, p tends to a steady value.
- Planetary waves carry the upwelling westward in a steadily widening zone.

Fig. 8. Plot of the imaginary part of (10.5) for $\Lambda = 150$ as a function of dimensionless time for the values of $\varepsilon = 0{\cdot}2, 1, 2, 4, 10$. Damped oscillations are evident for all values of ε, but only for the smallest value of ε does the period of the oscillation match the Kelvin wave period $2\pi\sqrt{/\Lambda\varepsilon}$, indicated by the solid lines at foot of graph for $\varepsilon = 0{\cdot}2, 1, 2$.

Thermocline Displacement: beta-plane



- Planetary waves carry the upwelling westward in a steadily widening zone
- In upwelling areas, H is on the order of 100m giving a displacement of 30 m
- Damping time is then on the order of a month

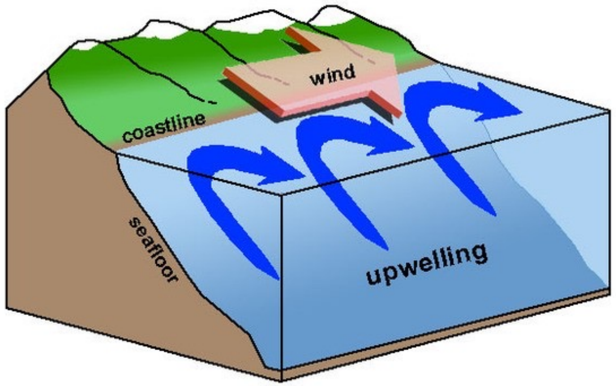


Fig. 9. Plot of the imaginary part of $\beta Hp/f\tau$ as a function of x away from the eastern boundary for $\Lambda = 150$, $\varepsilon = 0{\cdot}2$. The curves are plotted every 10 units of dimensionless time. The width of the upwelling zone can be seen to broaden with time and to asymptote to a value $\varepsilon/(1 + \varepsilon^2)$.

Thermocline Displacement: beta-plane

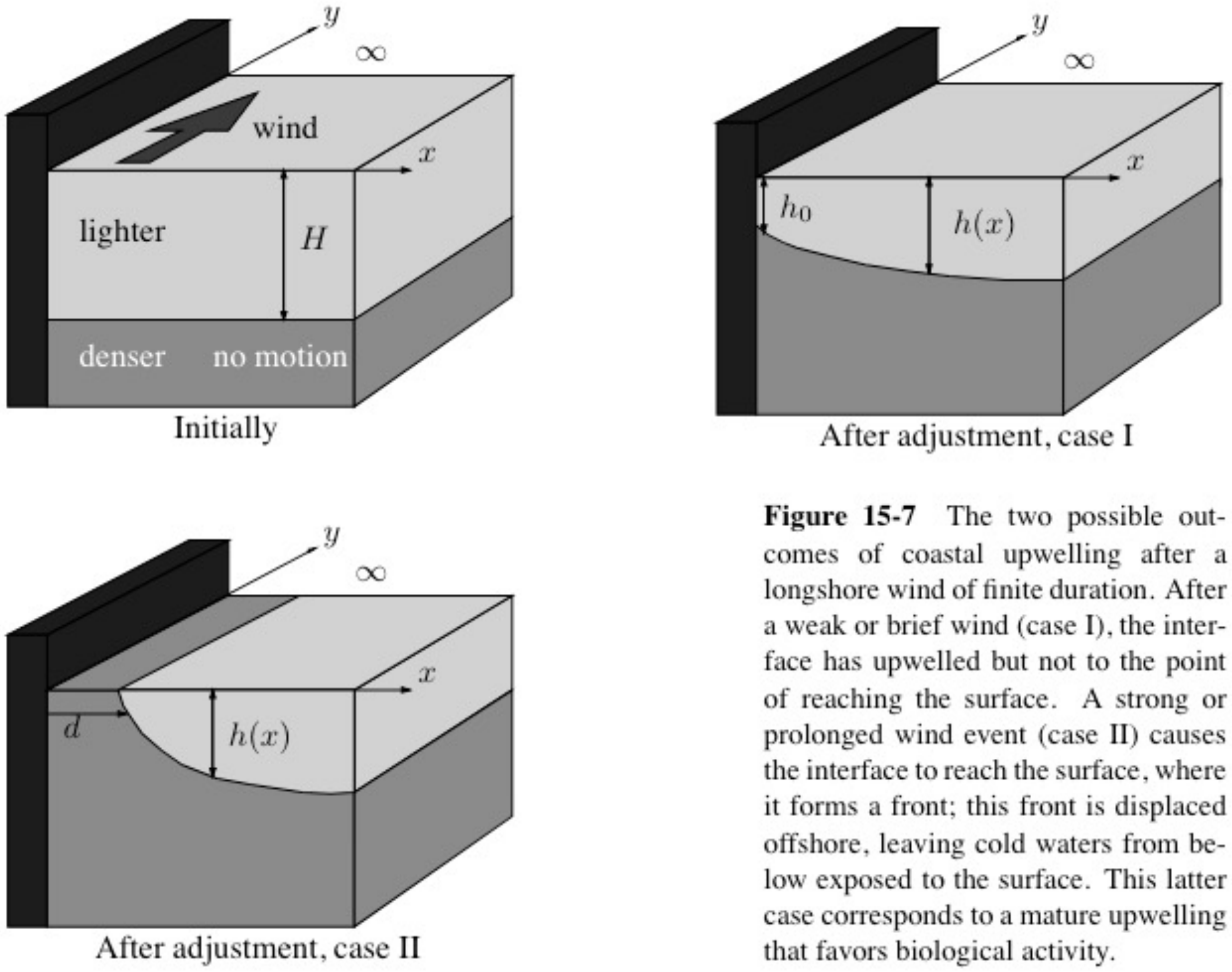


Figure 15-7 The two possible outcomes of coastal upwelling after a longshore wind of finite duration. After a weak or brief wind (case I), the interface has upwelled but not to the point of reaching the surface. A strong or prolonged wind event (case II) causes the interface to reach the surface, where it forms a front; this front is displaced offshore, leaving cold waters from below exposed to the surface. This latter case corresponds to a mature upwelling that favors biological activity.

- Related diagram to show the evolution of the upwelling where d is the distance the upwelling can travel from the coast on the beta-plane
- Image is not from the paper and only used to help visualize

Image credit: *Introduction to Geophysical Fluid Dynamics - Physical and Numerical Aspects*. (2011) (Vol. 101). Elsevier. https://doi.org/10.1016/C2009-0-00052-X

Recap for East-West

- Applied east-west wind stress to 2-layer ocean on beta-plane between fixed boundaries
- During spin-up, baroclinic flow increases linearly with time until non-dispersive Rossby wave (long wave) propagates from the eastern boundary, flow behind wave is Sverdrup solution
- Western boundary current builds up, thinning, until a small westward traveling wave crosses basin, then western boundary flow turns steady and thinning is reduced

Recap for North-South

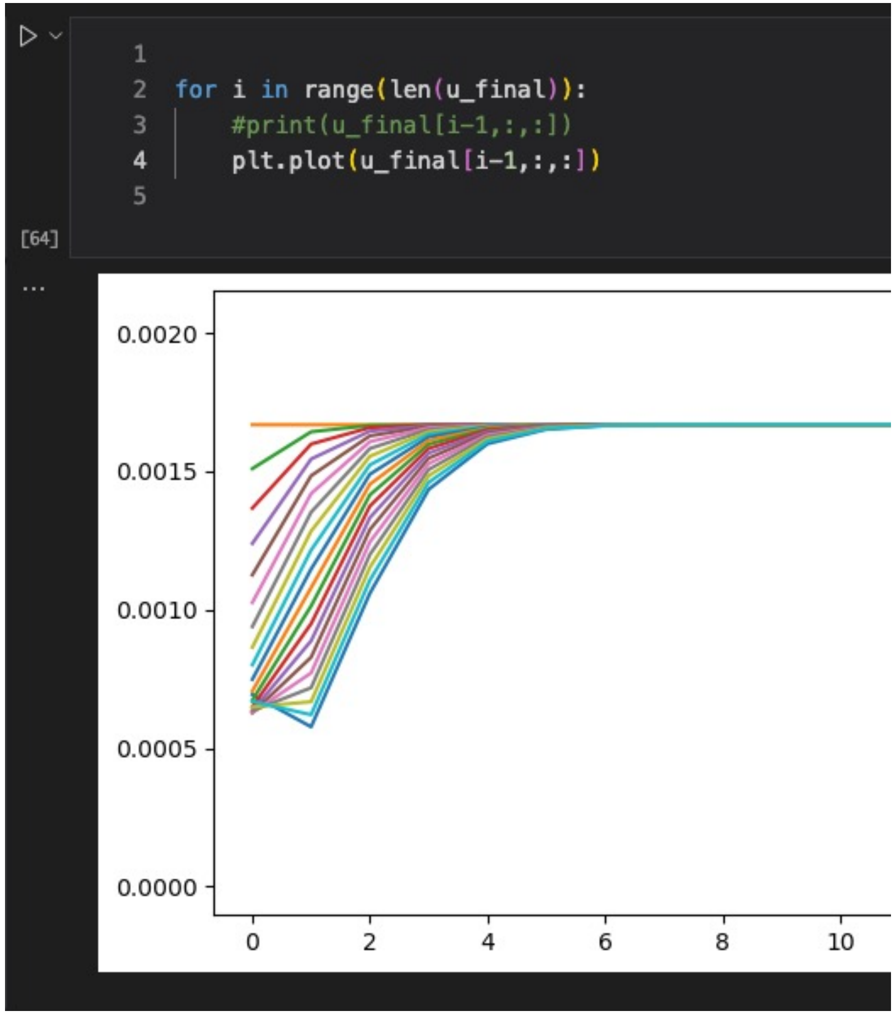
- Same idealized ocean starting at rest, longshore wind applied
- u in unperturbed in the interior (offshore) in both f-plane and beta-plane cases by Kelvin waves
- Beta-plane case allows the eastern boundary to grow and allow for u to relax to 0 – velocity can undergo reversal which can offset Ekman flux (above thermocline)

Recap for Thermocline

- On f-plane, mean upwelling occurs ¼ wavelength poleward from maximum equtorward wind stress because Kelvin waves are carrying energy poleward
- On f-plane, upwelling increasing linearly with time
- As the wave number l increases, the beta-plane solution approaches the f-plane solution near the coast
- Kelvin waves are damped on the beta-plane
- Upwelling is not restricted on the beta-plane

This is an Awesome paper

- A lot of great information packed into it
- The approximations and solutions have some inherent beauty, an opportunity to experience where math uncovers some of the hidden wonders of the universe
- Didn't have much time, but started to try to numerically solve some of the solutions myself



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ANY
QUESTIONS?