$$\boxed{\begin{array}{ll} x\text{-momentum (linear)} & \dfrac{\partial u}{\partial t} + U \dfrac{\partial u}{\partial x} - f v = -g \dfrac{\partial \eta}{\partial x} \\[2ex] y\text{-momentum (linear)} & \dfrac{\partial v}{\partial t} + U \dfrac{\partial v}{\partial x} + f u = -g \dfrac{\partial \eta}{\partial y} \end{array}} \quad \begin{array}{l} \Big| \cdot \dfrac{\partial}{\partial y} \\ - \\ \Big| \cdot \dfrac{\partial}{\partial x} \end{array}$$

where U is a constant or mean flow or advection

$$f = f_0 + \beta \cdot y$$

continuity (mass conservation) non-divergent $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = -\dfrac{1}{H}\dfrac{\partial \eta}{\partial t} = 0$

take y-derivative of x-momentum and subtract from x-derivative of y-momentum

$$\frac{\partial}{\partial t}\left(u_y - v_x\right) + U \frac{\partial}{\partial x}\left(u_y - v_x\right) - \left(\beta v\right) - \cancelto{0}{f\left(u_x - v_y\right)} = \cancelto{0}{-g\left(\eta_{xy} - \eta_{xy}\right)}$$

or

$$\boxed{\frac{\partial}{\partial t}\left(u_y - v_x\right) + U \frac{\partial}{\partial x}\left(u_y - v_x\right) - \beta v = 0 \qquad \text{vorticity equation}}$$

Any non-divergent velocity field allows stream functions ψ, that is

$$u = -\frac{\partial \psi}{\partial y} = -\psi_y \quad \text{and} \quad v = +\frac{\partial \psi}{\partial x} = \psi_x$$

The vorticity can thus be written in terms of ψ as

$$\frac{\partial}{\partial t}\left(-\psi_{yy} - \psi_{xx}\right) + U \frac{\partial}{\partial x}\left(-\psi_{yy} - \psi_{xx}\right) - \beta \psi_x = 0$$

or $\quad \dfrac{\partial}{\partial t}\left(\nabla^2 \psi\right) + U \dfrac{\partial}{\partial x}\left(\nabla^2 \psi\right) + \beta \psi_x = 0$

This is also a wave equation for $\psi = \psi(x, y, t)$ that allows solutions of the form

$$\psi = \psi_0 \exp[i(kx + ly - \omega t)]$$

for wave numbers (k, l) and frequency ω.

Note that

$$\frac{\partial}{\partial t}\psi = -i\omega\psi \qquad \frac{\partial}{\partial x}\psi = +ik\psi \qquad \frac{\partial}{\partial y}\psi = +il\psi$$

So the vorticity / wave equation

$$\frac{\partial}{\partial t}(\psi_{xx} + \psi_{yy}) + U\frac{\partial}{\partial x}(\psi_{xx} + \psi_{yy}) + \beta\psi_x = 0$$

$$-\cancel{i}\omega\,(i^2k^2 + i^2l^2) + U\cancel{i}k\,(i^2k^2 + i^2l^2) + \beta\cancel{i}k = 0$$

$$-\omega\,(-k^2 - l^2) + Uk\,(-k^2 - l^2) + \beta\cdot k = 0$$

$$\omega\,(k^2 + l^2) - Uk\,(k^2 + l^2) + \beta\cdot k = 0$$

$\Rightarrow$

$$\omega = \frac{-\beta k + Uk(k^2 + l^2)}{k^2 + l^2}$$

or

$$\boxed{\omega = \frac{-\beta k}{k^2 + l^2} + U\cdot k \qquad \text{Dispersion}}$$

For zonal waves (east-west propagating) $k^2 \gg l^2 \Rightarrow (k^2 + l^2) \approx k^2$

$$\boxed{\omega = \underbrace{-\frac{\beta}{k}}_{\text{Rossby Wave}} + \underbrace{U\cdot k}_{\text{non-dispersive "other" wave}} \qquad \Rightarrow c_p^{(x)} = \frac{\omega}{k} = -\frac{\beta}{k^2} + U \qquad c_g^{(x)} = \frac{\partial\omega}{\partial k} = +\frac{\beta}{k^2} + U}$$

The dispersion relation relates – always – wave periods to wavelength or frequency as a function of wavenumbers and physical system parameters, e.g.,

$$\omega = -\beta / k + U \cdot k$$

- The two components $-\beta / k$ and $+U \cdot k$ represent two different aspects of the physics that we identify by their system parameters β (planetary vorticity) and U (constant advection).
- The β-term recalls a Rossby wave we discussed earlier that had a dispersion $\omega = -\beta \cdot k \,/\, [k^2 + (1/L_D)^2]$ where $L_D = \sqrt{g \cdot H} / f$ that arose from $U = 0$ <u>and</u> $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{H}\frac{\partial \eta}{\partial t} = 0$ the non-divergent continuity
- Our $(-\beta / k)$ term is recovered for $k^2 \gg (1/L_D)^2$ or $L_D \cdot k \gg 1$ as the short wave limit of pure Rossby Waves that propagate their energy (via group velocity) to the East. By Ignoring the $\frac{\partial \eta}{\partial t}$ term in the continuity equation, we effectively "filtered out" (removed) the non-dispersive long Rossby waves.
- The $(U \cdot k)$ term is new and due to (linearized) advection.

The combination or sum of $(-\beta / k)$ and $(\mathcal{U} \cdot k)$ terms

$$\omega = -\beta/_k + \mathcal{U} \cdot k$$

offers many applications such as

- Weather Systems embedded within a zonal jet

  $$\boxed{\begin{array}{l}\text{Energy Propagation ALWAYS from West-to-East} \\ \text{group velocity } c_g^{(x)} = \dfrac{\partial \omega}{\partial k} = +\dfrac{\beta}{k^2} + \mathcal{U} > 0\end{array}}$$

  This is one reason that energetic weather system originate mostly from the west in both N. America and N. Europe.

- A "stationary" wave exists that has $c_p^{(x)} = 0$

  $$\boxed{c_p^{(x)} = \frac{\omega}{k} = -\beta/_{k^2} + \mathcal{U}}$$

  For $c_p^{(x)} = 0$ we have $\mathcal{U} = +\beta / k^2$

  or $k^2 = \beta / U$

- This gives us a new length scale $\boxed{L_\beta = \sqrt{\dfrac{U}{\beta}}}$

  For $U \sim 10$ m/s $\beta = 10^{-11}$ $m^{-1} s^{-1}$

  $L_\beta = \sqrt{10^{+12} m^2} = 1000$ km

- Even though the phase velocity is zero at $k = \sqrt{\beta / U}$

  the group velocity $c_g^{(x)} = +\beta/_{k^2} + \mathcal{U} = +2\,\mathcal{U}$

- I believe this impacts "Extreme Events" in climate change scenarios

- As the Jet stream velocity $\mathcal{U}$ becomes smaller as polar regions warm faster than mid-latitude regions, the "stationary" wave events propagate their energy at a slower rate $2 \cdot \mathcal{U}$.
  "Stationary" extreme events stay around longer and more pronounced flooding, droughts, heat waves, cold spells, etc. result.

- To the best of my current knowledge, applications to ocean phenomena such as Western Boundary Currents or Antarctic Circumpolar Current are not well published yet.