MAST-455/655 Geophysical Fluid Dynamics Mid-Term Exam Apr.-8, 2024 (Closed Book)

Consider modified linear shallow water equations applied to predict weather systems in the presence of a zonal flow U, a first approximation of the mid-latitude Jet Stream

(1)	$\partial_t u + U\,\partial_x u - f\,v = -g\,\partial_x \eta$	momentum balance in x (east-west)
(2)	$\partial_t v + U\,\partial_x v + f\,u = -g\,\partial_y \eta$	momentum balance in y (north-south)
(3)	$\partial_x u + \partial_y v = 0$	continuity or mass conservation

that describe a time-dependent velocity field (u,v) embedded within a known constant flow U on a rotating earth where the Coriolis parameter varies as $f=f_0+\beta y$.

1. [20 pts] Derive the vorticity equation and exploit the continuity equation (3) that is divergence-free and thus allows you to select a stream function $\psi=\psi\,(x,y,t)$ such that

$$u = -\partial_y \psi \quad \text{and} \quad v = +\partial_x \psi.$$

2. [30 pts] Solve the vorticity equation by seeking free wave solutions

$$\psi=\psi_0\, exp[\, i\, (\, kx+ly-\omega t\,)]$$

to derive a dispersion relation that relates frequency to wave numbers (k,l). [This gives $u=-il\psi$ and $v=+ik\psi$, so $u/v=-l/k$ which describes a clock-wise rotating (in time) current ellipse with a semi-major axis oriented perpendicular to the wavenumber vector (k,l).]

3. [30 pts] For each of the two cases below, sketch and interpret this dispersion equation for zonal waves ($k^2+l^2 \approx k^2$) with a graph $\omega=\omega(k)$ by looking for possible zero-crossing ($\omega=0$), minima, maxima, and/or asymptotic behaviors such as $k\to0$ and/or $k\to\infty$ and/or $k\to-\infty$. Find simple expressions for zonal phase and group velocities and comment on their directionality. Consider

   3.1 U=0 (no zonal current): What type of waves are these? Are they always dispersive? How many waves do you expect at each frequency?

   3.2 U=constant: Are these waves always dispersive? Can you distinguish short (k>>1) from long waves (k<<1)? How many waves do you expect at each frequency?

4. [20 pts] The case 3.2 above allows for a so-called “stationary wave” for which the phase velocity $c_p^{(x)} = 0$. What is the wavelength ($2\pi/k$) for this wave? What is the period of the corresponding Rossby wave (U=0)? What is its group velocity? Use typical mid-latitude values such as $\beta\sim10^{-11}$ m^{-1} s^{-1} and $U\sim10$ m/s. How does a stronger jet (U=50 m/s, say) change the results?