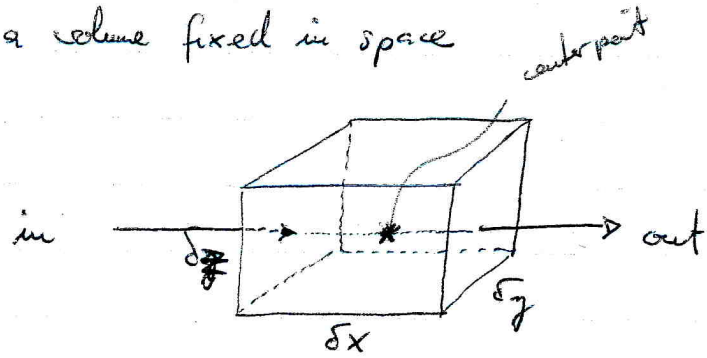
Equation #4:

Conservation of Mass

Consider a volume fixed in space

center point in the middle of box at delta(x)/2



rho * u in kg/m^3 * m/s)

is the mass flux (per unit area across the dy*dz area

in through $\delta y \delta z$: $\left[\rho u - \frac{1}{2}\frac{\partial(u\rho)}{\partial x}\delta x\right]\delta y\,\delta z$

out through $\delta y \delta z$: $\left[\rho u + \frac{1}{2}\frac{\partial}{\partial x}(u\rho)\,\delta x\right]\delta y\,\delta z$

in - out through $\delta y\,\delta z$ $\quad -\frac{\partial(u\rho)}{\partial x}\delta x\,\delta y\,\delta z$

in - out all sides

$$-\vec{\nabla}(\rho\vec{u})\,\delta x\,\delta y\,\delta z$$

As $\delta x\,\delta y\,\delta z = \delta V \to 0$ the rate of increase of mass per unit volume becomes

$$-\vec{\nabla}(\rho\vec{u}) = \frac{\partial\rho}{\partial t}$$

because ρ is $\frac{\text{mass}}{\text{unit volume}}$

or, finally

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{u}) = 0$$

mass conservation (no 4 equations for 5 variable

go to p. 22 to apply Boussinesq law.

Boussinesq Approximation

density of freshwater $\sim$ 1000 kg/m^3

ρ max in ocean $\sim$ 1040 kg/m^3 (Red Sea)

Hence it appears reasonable to assume

$$\rho = \rho_0 + \rho' \qquad \text{with } \rho' \ll \rho_0$$

$$40 \ll 1000$$

What about atmosphere where $\rho \in [0, 1]$ kg/m^3 ? show Engineering Toolbox graphs

$$\boxed{\begin{aligned} \rho &= \rho_0(z) + \rho'(x,y,z,t) \\ p &= p_0(z) + p'(x,y,z,t) \end{aligned} \qquad \text{with } \frac{\partial p_0}{\partial z} = -\rho_0 g}$$

This will effectively lead to the same approximations by considering density and pressure perturbations from a stably stratified density distribution $\rho_0(z)$ balanced by a hydrostatic pressure p_0

Consider Mass balance

$$\frac{\partial \rho}{\partial t} + \underbrace{\vec{\nabla}(\rho \vec{u})}_{\text{chain rule}} = 0 \qquad \text{with } \rho = \rho_0 + \rho'$$

$$\cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho'}{\partial t} + (\rho_0 + \rho')\,\vec{\nabla}\cdot\vec{u} + (\vec{u}\cdot\vec{\nabla})(\rho_0 + \rho') = 0$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \nabla\vec{u} + \frac{1}{\rho}(\vec{u}\cdot\nabla)\rho = \frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\vec{u} = 0 \qquad \text{or} \qquad \frac{D}{Dt}(\rho) + \rho\,\nabla\cdot\vec{u} = 0$$

Scaling Analysis:

$w\frac{\partial \rho_0}{\partial z}$ at most, but $w\frac{\partial \rho_0}{\partial z} \sim \rho' \frac{H}{T}\cdot\frac{1}{H} \sim \rho'/T$ because $\frac{\Delta \rho_0}{\Delta z} \sim$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla}\cdot\vec{u} + \rho' \vec{\nabla}\cdot\vec{u} + \cancel{(\vec{u}\cdot\vec{\nabla})\rho_0} + (\vec{u}\cdot\nabla)\rho' = 0$$

rho'/T	$\rho_0 U/L$	$\rho' U/L$	rho'/T	$\rho' U/L$
$\frac{\rho'}{T}$,	$\rho_0 \frac{1}{L}\frac{L}{T}$,	$\rho' \frac{1}{L}\frac{L}{T}$,	$\rho' \frac{H}{T}\cdot\frac{1}{H}$,	$\frac{L}{T}\cdot\frac{1}{L}\rho'$
$\frac{\rho'}{\rho_0}$	1	$\frac{\rho'}{\rho_0}$	$\frac{\rho'}{\rho_0}$	$\frac{\rho'}{\rho_0}$

w~H/T

and

d(rho0)/dz ~ rho'/H

$$\Rightarrow \quad \rho_0 \vec{\nabla}\cdot\vec{u} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{u} = 0$$

Mass Conservation
where rho' << rho0
(Boussinesq approximation)

New Mass Conservation (with Boussinesq Approximation) gives us

A New Scaling:

U/L scales the horizontal velocity variations in the horizontal
W/H scales the vertical velocity variations in the vertical

but BOTH are of the same order of magnitude

$$\boxed{U/L \sim W/H}$$

or

$$\boxed{W \sim U * H/L}$$

Here in components

Horizontal: $$\underbrace{\rho}_{(\rho_0+\rho')}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\boxed{+wf\cos\phi-f\sin\phi\, v}\right)=-\frac{\partial p}{\partial x}+\text{friction}$$

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but $\rho = \rho_0(z) + \rho'(x,y,z,t)$ with $\rho_0 \gg \rho'$

and $p = p_0(z) + p_0'(x,y,z,t)$ with $\frac{\partial p_0}{\partial z} = -\rho_0 g$

this gives

perturbation pressure as $\frac{\partial p_0}{\partial x}=0$ $p_0=p_0(z)$

Horizontal: $$\rho_0\left(\frac{\partial u}{\partial t}+\left[\vec{u}\cdot(\vec{u}\cdot\vec{\nabla})\vec{u}\right]+wf\cos\phi-fv\sin\phi\right)=-\frac{\partial p'}{\partial x}+\text{friction}$$

other horizontal component is similar, but the vertical component is more tricky

Vertical: $$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-f\cos\phi\, u\right)=-\frac{\partial p}{\partial z}-\rho g+\text{friction}$$

$$\boxed{(\rho_0+\rho')}\left(\quad — \;\; " \;\; —\quad\right)=\underbrace{-\frac{\partial p_0}{\partial z}-\frac{\partial p'}{\partial z}-\rho_0 g}_{=0 \text{ by definition}}-\rho' g+\text{friction}$$

rho0 >> rho'

we cannot neglect this term

→ buoyancy force $\frac{\rho'}{\rho_0}g$!

Hence our momentum balance becomes

$$\frac{D}{Dt}\vec{u} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho_0}\vec{\nabla}p' - \frac{\rho'}{\rho_0}\vec{g} + \vec{F}_{friction}$$

In the special case of $\cancel{\rho' = 0}$ (homogeneous fluid, uniform density

$$\rho' = 0$$

→ barotropic dynamics, no buoyancy forces

Otherwise $-\frac{\rho'}{\rho_0}\vec{g}$ represents the buoyancy force (reduced gravity)

→ baroclinic dynamics

Next: Scale these equations using $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$

that is $\frac{U}{L} \sim \frac{W}{H}$

or $W \sim U\frac{H}{L}$

or $W \ll U$ as $H \ll L$