Rossby Waves

Coriolis is no longer constant, but varies (linearily) with latitude.

show 4 slides from Chelton + Schlax (1996)

Non-dimensional momentum (Pedlosky, section 3.12)

$$Ro_T \frac{\partial u}{\partial t} + Ro\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - v(1+\beta y) = -\frac{\partial \eta}{\partial x} + Ek\frac{\partial^2 u}{\partial z^2}$$

Taylor Series Expansion of f around latitude phi0

where $\boxed{f = f(y) = 2\Omega \sin\phi_0 + 2\Omega\cos\phi_0 \frac{y^*}{R} + \mathcal{O}\left(\left(\frac{y}{R}\right)^2\right)}$

dimensional: thus $f(y) \cong f_0 + \beta_0 y^*$ $\quad y^* = L \cdot y$

non-dimensional f(y): $\boxed{\frac{f}{f_0} \cong 1 + \frac{\beta_0}{f_0}\cdot L\, y = 1 + \beta y}$ $\quad \beta = \frac{\beta_0 L}{f_0} \ll 1$

frictionless flow $\quad Ek \ll Ro_T \ll 1$

linear flow $\quad \boxed{\begin{array}{l} Ro \ll Ro_T \ll 1 \\ Ro_T \sim \beta \ll 1 \end{array}}$

Neglect "very small" nonlinear advection

Retain "small" local accelerations

$Ro = \frac{U}{fL}$ expand velocity and sealevel as $\boxed{\begin{array}{l} u' = u_0 + \beta u_1 + \mathcal{O}(\beta^2) \\ \eta' = \eta_0 + \beta \eta_1 + \mathcal{O}(\beta^2) \end{array}}$

$$\beta \frac{\partial(u_0 + \beta u_1)}{\partial t} + \cancel{Ro\left[\ldots\right]} - \underbrace{(v_0 + \beta v_1)}_{v}\underbrace{(1+\beta y)}_{f} = -\frac{\partial}{\partial x}(\eta_0 + \beta\eta_1) + \cancel{Ek\frac{\partial^2(u_0+\beta u_1)}{\partial z^2}} + \mathcal{O}(\beta^2)$$

non-dimensional

$(u, v) = U(u', v')$

$\eta = N_0 \eta'$

$$\boxed{N_0 = \frac{fUL}{g}} = \overbrace{\left(\frac{f^2}{gH}\right)\cdot L^2}^{(L/L_D)^2} \cdot Ro \cdot H$$

This scaling ensures that Coriolis and pressure gradient balance

x-momentum:

$$\boxed{\mathcal{O}(1): \quad -v_0 = -\frac{\partial \eta_0}{\partial x}} \xrightarrow{\text{dim.}} \left.\begin{aligned} v_0 &= +\frac{g}{f_0}\frac{\partial \eta_0}{\partial x} \\ u_0 &= -\frac{g}{f_0}\frac{\partial \eta_0}{\partial y} \end{aligned}\right\} \text{geostrophic} \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

$$\boxed{\mathcal{O}(\beta): \quad \frac{\partial u_0}{\partial t} - v_1 - v_0 y = -\frac{\partial \eta_1}{\partial x}} \longrightarrow \frac{\partial u_0}{\partial t} - f_0 v_1 - \beta_0 v_0 y = -g\frac{\partial \eta_1}{\partial x}$$

$$\frac{\partial v_0}{\partial t} + f_0 u_1 + \beta_0 u_0 y = -g\frac{\partial \eta_1}{\partial y}$$

dimensionally:

x-momentum:

$$v_1 = \left(\underbrace{\frac{\partial u_0}{\partial t}} - \underbrace{\beta_0 y v_0} + g\frac{\partial \eta_1}{\partial x}\right)\Big/ f_0 \qquad \text{known once } \eta_0, \eta_1 \text{ are known}$$

$$= -\frac{g}{f_0^2}\frac{\partial^2 \eta_0}{\partial y \partial t} - \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta_0}{\partial x} + \frac{g}{f_0}\frac{\partial \eta_1}{\partial x}$$

y-momentum

$$u_1 = -\frac{g}{f_0^2}\frac{\partial^2 \eta_0}{\partial x \partial t} + \frac{\beta_0 g}{f_0^2} y \frac{\partial \eta_0}{\partial y} - \frac{g}{f_0}\frac{\partial \eta_1}{\partial y} \qquad \text{known once } \eta_0, \eta_1 \text{ are known}$$

d/dy (x-momentum)
minus
d/dx (y-momentum

$$\boxed{\begin{aligned} \frac{\partial v_1}{\partial y} &= -\frac{g}{f_0^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta_0}{\partial y^2}\right) - \cancel{\frac{\beta_0 g}{f_0^2} y \frac{\partial^2 \eta_0}{\partial x \partial y}} - \frac{\beta_0 g}{f_0^2}\frac{\partial \eta_0}{\partial x} + \cancel{\frac{g}{f_0}\frac{\partial^2 \eta_1}{\partial y^2 \partial x}} \\ \frac{\partial u_1}{\partial x} &= -\frac{g}{f_0^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta_0}{\partial x^2}\right) + \cancel{\frac{\beta_0 g}{f_0^2} y \frac{\partial^2 \eta_0}{\partial y \partial x}} - \cancel{\frac{g}{f_0}\frac{\partial^2 \eta_1}{\partial x^2 \partial y}} \end{aligned}}$$

add to get divergence to derive vorticity

Continuity:

$$\frac{\partial \eta_0}{\partial t} + H\boxed{\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right)} = 0 \longrightarrow \frac{\partial \eta_0}{\partial t} - \underbrace{\frac{gH}{f_0^2}}_{a^2 \equiv L_R^2 \equiv gH/f_0^2}\frac{\partial}{\partial t}\left(\frac{\partial^2 \eta_0}{\partial x^2} + \frac{\partial^2 \eta_0}{\partial y^2}\right) - \beta_0 \frac{gH}{f_0^2}\frac{\partial \eta_0}{\partial x} = 0$$

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$$\underbrace{\frac{\partial \eta_0}{\partial t}}_{\text{vortex tube stretching}} - a^2 \underbrace{\frac{\partial}{\partial t}\left(\nabla^2 \eta_0\right)}_{\text{relative vorticity of } O(1) \text{ geostrophic flow}} - \underbrace{\beta_0 a^2 \frac{\partial \eta_0}{\partial x}}_{\text{vortex tube stretching by planetary } \beta \text{ (ambient vorticity) background}} = 0$$

a is the Rossby radius of deformation $a^2 = gH/f_0^2$

Skip

This is the Order(beta) vorticity equation for the Order(1) sea level eta0.

Recall that the Order(1) dynamics are geostrophic, but we now have a new and third equation for the three Order(1) variables u0, v0, and eta0.

Note that the above equation is a wave equation …

… to give us Rossby waves:

Try solutions of the form eta0 ~ exp[i(kx+ly-wt)]*