$$\boxed{\underbrace{\frac{\partial^2 p}{\partial x^2} + \frac{2\beta_0 H}{f_0 d}\frac{\partial p}{\partial x}}_{\text{western boundary}} = \overbrace{-\frac{d}{2}\frac{\partial \tau^{(x)}}{\partial y}}^{\text{Sverdrup Interior}}}$$

VORTICITY EQUATION

(1) Sverdrup Interior

$$\frac{\partial p_J}{\partial x} = -\frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}$$

known wind-forcing:
wind-stress curl

$$p_J(x,y) = -\frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y} \cdot x + P_1(y)$$

(2) Western Boundary

$$\frac{\partial^2 p_b}{\partial x^2} + \alpha \frac{\partial p_b}{\partial x} = 0$$

$$\boxed{\alpha = \frac{2\beta_0 H}{f_0 d}}$$

new length scale:
width of boundary current

$$p_b(x,y) = P_2(y) + P_3(y)\, e^{-\alpha x}$$

Matching conditions

$$\lim_{x_p \to \infty} p_b(x_p, y) = P_2(y)$$

$$\lim_{x_J \to 0} p_J(x_J, y) = P_1(y)$$

Δ $\quad P_1(y) = P_2(y)$ will ensure matching solutions

$$p_J(x,y) = P_1(y) - \frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\, x$$ no boundary condition applied yet

$$p_b(x,y) = P_1(y) + P_3(y)\, e^{-\alpha x}$$ no boundary condition applied yet

Merged Solution valid for both Sverdrup interior and western boundary

$$\boxed{p(x,y) = \underbrace{P_1(y)}_{\text{matching}} - \underbrace{\frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\, x}_{\text{small within western boundary layer}} + \underbrace{P_3(y)\, e^{-\alpha x}}_{\text{small outside western boundary}}}$$

Now apply boundary conditions :

eastern side $x = L_1$ $\quad u = -\frac{1}{\rho_0 f}\boxed{\frac{\partial p}{\partial y} = 0}$:

as exponential goes to zero at eastern boundary

$$\boxed{\frac{\partial p}{\partial y}} = \left.\frac{\partial P_1}{\partial y}\right|_{x=L_1} - \frac{f_0}{\beta_0 H}\frac{\partial^2 \tau^{(x)}}{\partial y^2} L_1 + \left.\frac{\partial P_3}{\partial y}\, \underbrace{e^{-\alpha x}}_{0}\right|_{x=L_1} = 0 \Big|_{x=L_1}$$

$$\Rightarrow \left.\frac{\partial P_1}{\partial y}\right|_{x=L_1} = + \frac{f_0 L_1}{\beta_0 H}\frac{\partial^2 \tau^{(x)}}{\partial y^2}$$

or

$$\boxed{\left.P_1(y)\right|_{x=L_1} = P_{10} + \frac{f_0 L_1}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}}$$

western side $x = 0$ $\quad u = -\frac{1}{\rho_o f} \boxed{\frac{\partial p}{\partial y} = 0}$:

$$\left.\frac{\partial P_1}{\partial y}\right|_{x=0} + \left.\frac{\partial P_3}{\partial y} e^{-\kappa x}\right|_{x=0} = 0$$

$$\left.\frac{\partial P_1}{\partial y}\right|_{x=0} + \left.\frac{\partial P_3}{\partial y}\right|_{x=0} = 0$$

$$P_1(y) + P_3(y) = P_{3o} = \text{constant} \quad @ \; x = 0$$

$$= \Psi(x=0, y) \cdot \rho_o f$$

where

$\Psi(x,y)$ is a stream function $\frac{\partial \psi}{\partial y} = -u$, $\frac{\partial \psi}{\partial x} = v$

so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Note that for $\boxed{\text{geostrophy} \quad u = -\frac{1}{\rho_o f} \frac{\partial p}{\partial y} \; , \; v = +\frac{1}{\rho_o f} \frac{\partial p}{\partial x}}$

$$\Psi = \frac{p}{\rho_o f} + \text{const}$$

As the constant does not matter, we can set $P_{3o} = \Psi(x=0,y) \cdot \rho_o f = 0$

Thus $\boxed{P_3(y) = -P_1(y)}$

and thus

$$p(x,y) = P_{10} + \frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\underbrace{(L_1 - x)}_{\to 0 \text{ as } x \to L_1}$$

$$- \left(P_{10} + \frac{f_0}{\beta_0 H} L_1 \frac{\partial \tau^{(x)}}{\partial y}\right)\underbrace{e^{-\alpha x}}_{\to 0 \text{ as } x \to L_1}$$

Now evaluate this pressure field at the eastern boundary $x = L_1$

$$p(x = L_1, y) = P_{10} = \rho_0 f\, \Psi(x = L_1, y)$$

but we also want to impose $\displaystyle\int_0^{L_1} v\, dx = 0$

This "defines" the ocean gyre

we have prescribe a closed basin OR a closed gyre bounded by $\frac{\partial \tau^{(x)}}{\partial y} =$ in North + South

or

$$0 = \int_0^{L_1} \frac{\partial \Psi}{\partial x} dx = \Psi(x = L_1) - \underbrace{\Psi(x = 0)}_{\text{we already set this to } 0 \text{ on western boundary}}$$

fix $\Psi(x = L_1) = 0$

fix $P_{10} = 0$

fix $$\boxed{p(x,y) = \frac{f_0 L_1}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\left[1 - \frac{x}{L_1} - e^{-\alpha x}\right]}$$