Sverdrup Interior

$$\frac{\partial^2 p}{\partial x^2} + \frac{2\beta_0 H}{f_0 d}\frac{\partial p}{\partial x} = -\frac{d}{2}\frac{\partial \tau^{(x)}}{\partial y}$$

western boundary

VORTICITY EQUATION

(1) Sverdrup Interior

$$\frac{\partial p_J}{\partial x} = -\frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}$$

$$p_J(x,y) = -\frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\cdot x + P_1(y)$$

(2) Western Boundary

$$\frac{\partial^2 p_b}{\partial x^2} + \alpha\frac{\partial p_b}{\partial x} = 0 \qquad \alpha = \frac{2\beta_0 H}{f_0 d}$$

$$p_b(x,y) = P_2(y) + P_3(y)\, e^{-\alpha x}$$

Matching conditions

$$\lim_{x_p \to \infty} p_b(x_p, y) = P_2(y)$$

$$\lim_{x_J \to 0} p_J(x_J, y) = P_1(y)$$

$P_1(y) = P_2(y)$ will ensure matching solutions

$$p_S(x,y) = P_1(y) - \frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\, x$$

no boundary condition applied yet

$$p_b(x,y) = P_1(y) + P_3(y)\, e^{-\alpha x}$$

no boundary condition applied yet

Merged Solution valid for both Sverdrup interior and western boundary

$$\boxed{p(x,y) = \underbrace{P_1(y)}_{\text{matching}} \underbrace{- \frac{f_0}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}\, x}_{\text{small within western boundary layer}} \underbrace{+ P_3(y)\, e^{-\alpha x}}_{\text{small outside western boundary}}}$$

Now apply boundary conditions :

eastern side $x = L_1$ $\quad u = -\frac{1}{\rho_0 f}\frac{\partial p}{\partial y} = 0$:

$$\frac{\partial p}{\partial y} = \left.\frac{\partial P_1}{\partial y}\right|_{x=L_1} - \frac{f_0}{\beta_0 H}\frac{\partial^2 \tau^{(x)}}{\partial y^2}\, L_1 + \left.\underbrace{\frac{\partial P_3}{\partial y}\, e^{-\alpha x}}_{0}\right|_{x=L_1} = 0\,\Big|_{x=L_1}$$

$$\left.\frac{\partial P_1}{\partial y}\right|_{x=L_1} = +\frac{f_0}{\beta_0 H}L_1\frac{\partial^2 \tau^{(x)}}{\partial y^2} \quad \text{or} \quad \boxed{\left.P_1(y)\right|_{x=L_1} = P_{10} + \frac{f_0 L_1}{\beta_0 H}\frac{\partial \tau^{(x)}}{\partial y}}$$