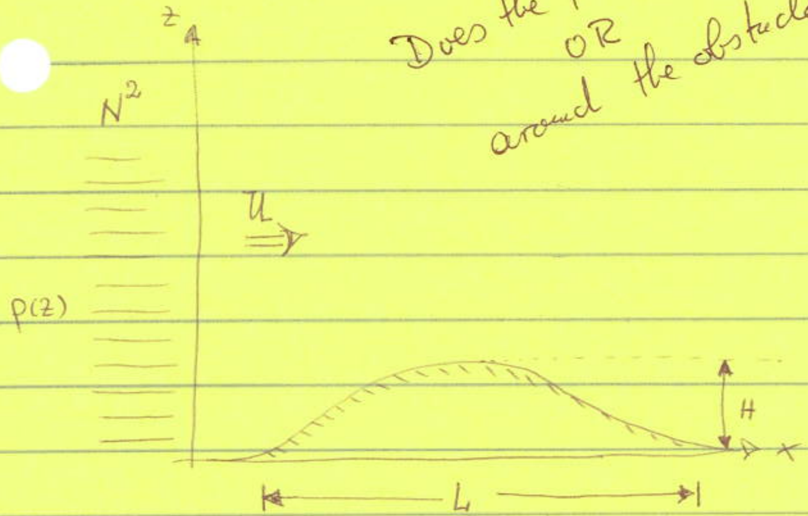
Does the flow go over OR around the obstacle?

in a density stratified fluid that has both vertical AND horizontal variations

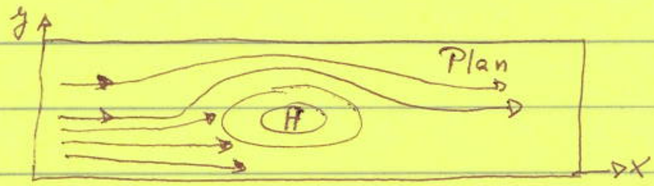


$T \sim L/U$ horizontal scale

$\Delta z \sim W \cdot T \sim W \cdot L/U$ vertical scale

but this now causes density changes

$$\boxed{\Delta\rho \approx \left(\frac{\partial\rho}{\partial z}\right) \cdot (\Delta z) \sim \left(\frac{N^2\rho_0}{g}\right) \cdot (W L/U)}$$



These density variations correspond to pressure variations

Note that H is a scale height related to stratification

Scale for vertical pressure differences:

$$\boxed{\Delta p \sim gH \cdot \Delta\rho \sim gH\,\frac{N^2\rho_0}{g}(WL/U)}$$

(from hydrostatic $\frac{\Delta p}{\Delta z} = \rho_0' g$ or $\Delta p \sim \Delta\rho \cdot g H$) $\Delta z \sim H$

$= H \boxed{\frac{g}{\rho_0}\frac{\Delta\rho}{H}} \cdot \rho_0 \frac{WL}{U} = \Delta\rho\, g \frac{W}{U/L}$

N^2

Scale for horizontal pressure differences:

scale $\boxed{\Delta p \sim \rho_0 U^2}$ ageostrophic velocity $\frac{\partial p}{\partial x} = \rho_0\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \sim \rho_0 \frac{U^2}{L}$

this gives

$$\boxed{\rho_0 U^2} \sim \boxed{\rho_0 N^2 H (WL/U)}$$

or $\frac{U^2}{N^2H^2} \sim \frac{W/H}{U/L} = \frac{W/H}{U/L}$

$N^2 = -\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}$ $\left(\frac{1}{s^2}\right)$

or $\left(\frac{U}{NH}\right)^2 \sim \frac{W/H}{U/L}$

note that $(NH)^2 = -\frac{g}{\rho_0}\frac{\partial\rho}{\partial z} \cdot H^2$

$\approx -g\frac{\Delta\rho}{\rho_0} H = c_i$

vertical

$$\Delta p \sim g \cdot \Delta \rho \cdot \Delta z \qquad \left(\text{from hydrostatic perturbation pressure } \frac{\partial p}{\partial z} = -\rho' g\right)$$

$$\sim g \, \Delta \rho \, \frac{W}{U/L} \qquad \Delta z \sim W \cdot T \sim W \cdot L/U$$

horizontal

$$\Delta p \sim \rho_0 U^2 \qquad \text{ageostrophic velocity } \frac{\partial p}{\partial x} = \rho_0 \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)$$

$$\sim \rho_0 U^2 / L$$

$$\boxed{\Delta p_{vertical}} = \boxed{\Delta p_{horizontal}}$$

$$\boxed{g \cdot \Delta \rho \frac{W}{U/L}} = \boxed{\rho_0 U^2}$$

or

$$U^2 = \boxed{g \frac{\Delta \rho}{\rho_0} \frac{W}{U/L} \cdot \frac{\Delta z}{\Delta z}} \qquad N^2 \approx \frac{-g}{\rho_0} \frac{\Delta \rho}{\Delta z}$$

$$\Delta z \sim H \sim W \cdot L/U$$

$$\approx \boxed{N^2 \cdot \frac{W}{U/L} \cdot H} \cdot \frac{H}{H}$$

$$\underbrace{\frac{U^2}{(NH)^2}}_{Fr^2} \sim \underbrace{\frac{W/H}{U/L}}_{\text{stiffness}} \sim 1 \qquad \text{as } \frac{W}{U/L} \sim H$$

internal wave phase speed:

$$\text{Note that } (NH)^2 = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \cdot H^2 \approx g \frac{\Delta \rho}{\rho_0} H = C_{internal}$$

kinetic energy

Internal Froude Number $Fr^2 = \left(\dfrac{U}{NH}\right)^2 = \dfrac{W/H}{U/L}$

potential energy due to buoyancy

So if

$\boxed{Fr \leq 1}$ stratification forces fluid $\Delta z \leq H$ vertically

or $W/H \leq U/L$

and a portion of the fluid will go around (rather than over) the obstacle

→ recall Taylor-Proudman theorem

$\boxed{\text{strong stratification} \rightarrow \text{large } N \rightarrow \text{small Froude Number} \rightarrow \underline{\text{stiff}} \text{ fluid}}$

Also compare

$$\frac{U}{N \cdot H} = Fr \longleftrightarrow Ro = \frac{U}{L \cdot f}$$

length scale

frequency

For QG dynamics we can expand
(almost geostrophic)

QG "Quasi-Geostrophic"

$$u = u_0 + Ro\, u_1 + \ldots$$
$$v = v_0 + Ro\, v_1 + \ldots$$

geostrophic velocity

Horizontal divergence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \underbrace{\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}}_{\substack{\text{geostrophic} \\ = 0}} + \underbrace{Ro}_{O(Ro)} \underbrace{\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right)}_{O(U/L)} + \cdots$$

$$= O\left(Ro \frac{U}{L}\right)$$

Vertical divergence balanced by Horizontal divergence

$$\frac{\partial w}{\partial z} \sim \frac{W}{H} = O\left(Ro \frac{U}{L}\right)$$

STIFF means $\frac{W}{H} \ll \frac{U}{L}$

So $\frac{W/H}{U/L} = O(Ro)$

~~stiff fluid~~ for $Ro \ll 1$ (Taylor Proudman)

$\boxed{\text{analogous}}$

But $\frac{W/H}{U/L} \sim \left(\frac{U}{NH}\right)^2 \sim Fr^2$

~~stiff fluid~~ for $Fr \ll 1$

Thus in a strongly stratified flow (N large) and $Fr^2 \ll 1$,

we can expect feature similar to Taylor columns [particles move along f/h contours]

and horizontally blocked flows that we encountered

before for $Ro \ll 1$

How to scale pressure?

geostrophic scaling, page-26a, Lecture-5

$P_v \sim \Delta\rho\, g H$ from vertical momentum $\frac{\partial p}{\partial z} = -g\rho'$

geostrophic $P_h^{(g)} \sim \rho_0 L U f$ from horizontal momentum $f v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$

inertial $P_h^{(i)} \sim \rho_0 U^2$ from $\frac{\partial p}{\partial x} = \rho_0\left(u\frac{\partial x}{\partial x} + v\frac{\partial x}{\partial y}\right)$

inertial scaling

thus $U \sim L \cdot f$

1. $$\frac{P_{vertical}}{P^{(g)}_{horizontal}} = \frac{\Delta\rho\, g H}{\rho_0 L U f} = \frac{\Delta\rho}{\rho_0} g H \cdot \frac{U}{Lf} \cdot \frac{1}{U^2}$$

$$= \frac{g}{\rho_0}\frac{\Delta\rho}{H}\frac{H^2}{g} \cdot \frac{U}{Lf} \cdot \frac{1}{U^2}$$

$$= \frac{N^2 H^2}{U^2} \cdot \frac{U}{Lf} = \boxed{\frac{R_o}{F_r^{\,2}}}$$

$N^2 \sim \frac{\Delta\rho}{\rho_0}\frac{g}{H}$

$(NH)^2 \sim \frac{\Delta\rho}{\rho_0} g H$

2. $$\frac{P_{vertical}}{P^{(i)}_{horizontal}} = \frac{\Delta\rho\, g H}{\rho_0 U^2} = \frac{(NH)^2}{U^2} = \boxed{\frac{1}{F_r^{\,2}}}$$

The scale for $P^1_{horizontal} \sim P^2_{horizontal}$ only for $R_o \sim 1$

3. $$\frac{P^{(g)}_{horizontal}}{P^{(i)}_{horizontal}} = \frac{\rho_0 L U f}{\rho_0 U^2} = \frac{fL}{U} = \boxed{\frac{1}{R_o}}$$

often though $R_o \ll 1$

or hydraulic pressure (nonlinear)

$\ll$

geostrophic pressure (linear

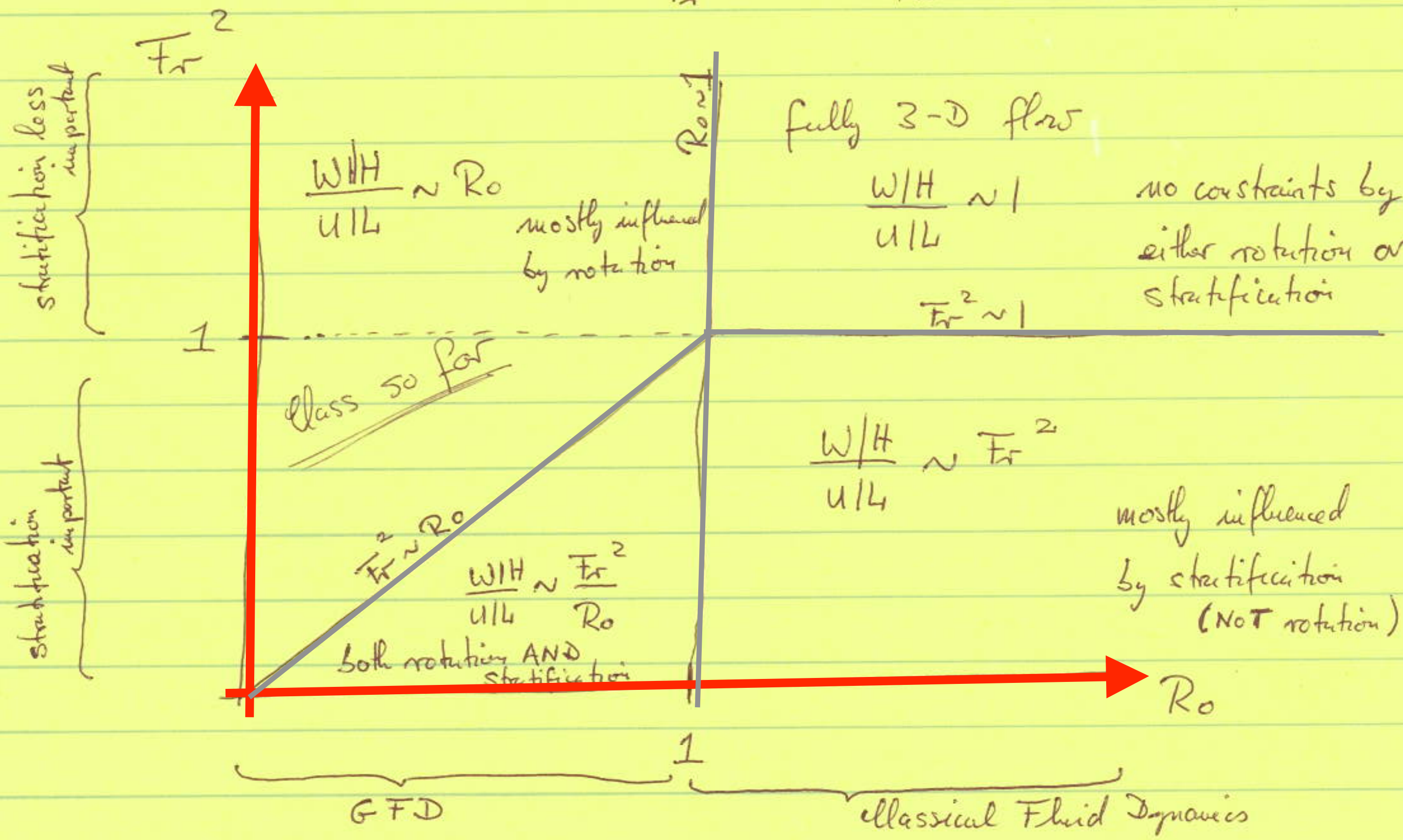
$$P_V = P_{H_g} : \Delta\rho g H = \rho_0 L \cdot U \cdot f$$

$$\text{or} \quad \frac{\Delta\rho}{\rho_0} \frac{g}{H} \cdot H^2 = L \cdot U \cdot f$$

$$\frac{N^2 H^2}{U^2} = \frac{L \cdot f}{U}$$

$$\frac{U^2}{(NH)^2} = \frac{U}{Lf}$$

$$Fr^2 = Ro$$



$$\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{\sim U/L} + \underbrace{\frac{\partial w}{\partial z}}_{\sim W/H} = 0$$