Thermal Wind Balance or Geostrophy for Stratified Flows

Stratified Geostrophic Dynamics

$$Ro \ll 1, \quad Ro_T \ll 1, \quad Ek \ll 1, \quad \boxed{Fr \leq 1, \quad \rho' \neq 0, \text{ but } \frac{\rho'}{\rho_0} \ll 1}$$

x-momentum: $$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

y-momentum: $$+fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad \Big| \frac{\partial}{\partial z}$$

z-momentum: $$\frac{\partial p}{\partial z} = -\rho' g \quad \Big| \frac{\partial}{\partial y}$$

$$\vec{\nabla} \times \vec{u} = \begin{pmatrix} \boxed{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}} \\ \boxed{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$$

$$\frac{\partial}{\partial y}(z\text{-mom}) - \frac{\partial}{\partial z}\, y\text{-mom}: \quad \cancel{\frac{\partial^2 p}{\partial y \partial z}} + g\frac{\partial \rho'}{\partial y} - f\frac{\partial(u\rho_0)}{\partial z} - \cancel{\frac{\partial^2 p}{\partial z \partial y}} = 0$$

(horizontal component of the vorticity vector $\vec{\nabla} \times \vec{u}$)

$$g\frac{\partial \rho'}{\partial y} - \boxed{fu\frac{\partial \rho_0}{\partial z}} - f\rho_0\frac{\partial u}{\partial z} = 0$$

$$g\frac{\Delta\rho}{L} \;;\; \boxed{fU\frac{\Delta\rho}{H}} \;;\; f\rho_0\frac{U}{H} \quad \Big| \cdot \frac{L}{\rho_0 g}$$

$$\frac{\Delta\rho}{\rho_0} \;;\; \boxed{\boxed{\frac{\Delta\rho}{\rho_0}}\frac{fLU}{gH}} \ll \frac{fLU}{gH}$$

small << large

This "scaling" results in a new horizontal length scale that is the

$$\boxed{\frac{\Delta\rho}{\rho_0} \sim \frac{fLU}{gH}}$$

Internal Rossby Radius:

gives $L^2 \sim \frac{\Delta\rho}{\rho_0}\frac{gH}{f^2} = L_D$ with $U \sim L \cdot f$

SKIP (redundant)

$$\nabla \times \vec{u} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \end{pmatrix}$$

$$\frac{\partial (z\text{-mom})}{\partial y} - \frac{\partial}{\partial z}(y\text{-mom}) : \cancel{\frac{\partial p}{\partial y \partial z}} + g\frac{\partial \rho}{\partial y} - \frac{\partial}{\partial z}(f\rho_0 u) - \cancel{\frac{\partial p}{\partial y \partial z}} = 0$$

$$\frac{\partial \rho}{\partial y} = \frac{f}{g}\frac{\partial}{\partial z}(\rho_0 u)$$

$$\frac{\partial \rho}{\partial y} = \frac{f u}{g}\frac{\partial \rho_0}{\partial z} + \frac{f}{g}\rho_0 \frac{\partial u}{\partial z}$$

$$\frac{\cancel{\Delta\rho}}{L}, \quad \frac{fU}{g}\frac{\Delta\rho}{H}, \quad \frac{f}{g}\frac{\rho_0}{\Delta\rho}\frac{U}{H}$$

$$\frac{\cancel{*}}{L}, \quad \frac{fU}{gH} \qquad \frac{fU}{gH}\frac{\rho_0}{\Delta\rho}$$

and $\frac{\Delta\rho}{\rho_0} \sim \frac{fUL}{gH}$

$$1 \qquad \frac{fUL}{gH} \qquad \frac{fUL}{gH}\frac{\rho_0}{\Delta\rho}$$

$$\frac{\frac{\Delta\rho}{\rho_0} gH}{fUf} \sim L^2$$

$$L^2 \sim \frac{\frac{\Delta\rho}{\rho_0} gH}{f^2} = L_D^2$$

$$\frac{\Delta\rho}{\rho_0} \qquad \frac{\Delta\rho}{\rho_0}\frac{fUL}{gH} \ll \frac{fUL}{gH}$$

gives $$\boxed{\frac{\partial u}{\partial z} = \frac{g}{\rho_o f}\frac{\partial \rho'}{\partial y}}$$ "thermal wind balance"

geostrophy for a density stratified fluid

or in vector form $$\frac{\partial \vec{u}_H}{\partial z} = -\frac{g}{\rho_o f}\vec{k} \times \nabla_H \rho$$

or $$\Delta \vec{u}_H = -\frac{g}{\rho_o f}\Delta z\, \vec{k} \times \nabla_H \rho$$

discuss examples

(1) Gulf Stream

(2) Nares Strait

1. Gives vertical shear of horizontal velocities only;

2. vertical velocity shear proportional to horizontal density gradients;

3. to estimate velocity, need a "reference" depth of "known" or "zero" motion"

4. bread & butter of observational physical oceanography since ~1920