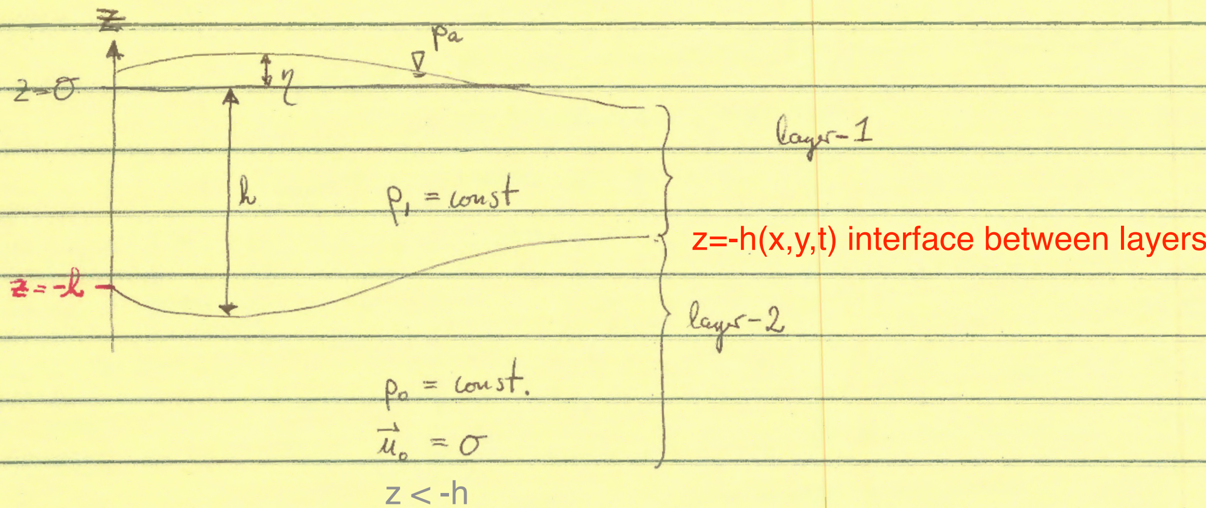
Layered Models (Cushman-Roisin, Chapter-12)

Reduced Gravity Model



Apply hydrostatic balance to layer-1:

$$\frac{\partial p_1}{\partial z} = -\rho_1 g \;\longrightarrow\; \int_z^{\eta} \frac{\partial p_1}{\partial z}\, dz = -\rho_1 g \int_z^{\eta} dz$$

Upper Layer Pressure: $$\boxed{\underbrace{p_1(x,y,\eta,t)}_{p_a} - p_1(x,y,z,t) = -\rho_1 g(\eta - z)}$$

So

$$\boxed{p_1(x,y,z,t) = p_a + \rho_1 g(\eta - z)}$$

z = -h: and $p_1(x,y,\underset{z=-h}{\wedge},t) = p_a + \rho_1 g(\eta + h) \equiv p_i$ — pressure at the interface separating layer-1 and layer-2

In the lower layer z< -h

$$\int_z^{-h} \frac{\partial p_0}{\partial z}\, dz = -\rho_0 g \int_z^{-h} dz \longrightarrow \underbrace{p_0\Big|_{z=-h}}_{p_i} - p_0(x,y,z,t) = -\rho_0 g(-h - z)$$

$|z| \geq h > 0$ in lower layer

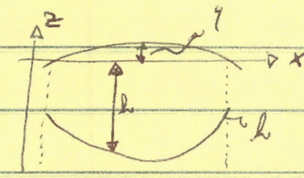
So

$$p_0(x,y,z,t) = p_i - \rho_0 g(h+z) = \overbrace{p_a + \rho_1 g(\eta + h)}^{p_i} - \rho_0 g(h+z)$$

$$\nabla_H \cdot p_0 = \underset{0}{\cancel{\nabla \cdot p_a}} + \rho_1 g \nabla \cdot \eta - (\rho_0 - \rho_1) g \nabla_H h \; \boxed{= 0 \quad \text{because } \vec{u}_0 = 0}$$

$\Delta\rho = \rho_0 - \rho_1 >$

So $$\boxed{\nabla_H \eta = \frac{\rho_0 - \rho_1}{\rho_1} \nabla_H h}$$ or $$\boxed{\eta = \frac{\Delta\rho\, h}{\rho_1}} + \text{const.}$$



but $\frac{\Delta\rho}{\rho} \ll 1$ thus $\frac{\Delta\eta}{\Delta h} \ll 1$ also

η large
so h large

Now return to upper layer

$$\nabla_H p_1(x,y,z,t) = \cancel{\nabla_H p_a}^{\,0} + \rho_1 g \boxed{\nabla_H \eta}$$ upper layer pressure gradient

$$= \cancel{\rho_1} g \boxed{\frac{\Delta\rho}{\cancel{\rho_1}} \nabla_H h}$$ expressed as gradient of interface height h

$$= g\, \Delta\rho\, \nabla_H h$$

or

$$\boxed{\frac{\nabla_H p_1}{\rho_0} = g \frac{\Delta\rho}{\rho_0} \nabla_H h = g' \nabla_H h}$$ where $g' = \frac{\Delta\rho}{\rho_0} g$

is the reduced gravity

This is the dynamically active horizontal pressure gradient and

the horizontal momentum balance ~~becomes~~ in the upper layer becomes

$$\boxed{\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + \cancel{w_1 \frac{\partial u_1}{\partial z}}^{\,0} - f v_1 = -g' \frac{\partial h}{\partial x}}$$

no mixing

If the interface $h = h(x,y,t)$ does not $\boxed{\text{"leak"}}$ then we can treat the upper layer as a "shallow water" layer and use continuity as

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_1 h) + \frac{\partial}{\partial y}(v_1 h) = 0}$$

Then all prior "shallow water" results of the depth-averaged (barotropic) dynamics apply to layer-averaged (baroclinic) dynamics with the replacement

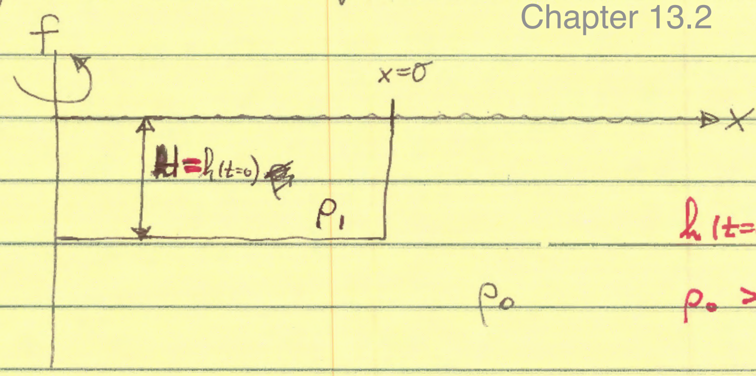
$$\boxed{g \longrightarrow \frac{\Delta\rho}{\rho_0} g}$$

reduced gravity

SHOW "geostrophic adjustment" movie again

Cushman-Roisin (1994)
Chapter 13.2

For $\frac{\partial}{\partial y} \rightarrow 0$



$h(t=0) = H$

$\rho_0 > \rho_1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu = 0$$

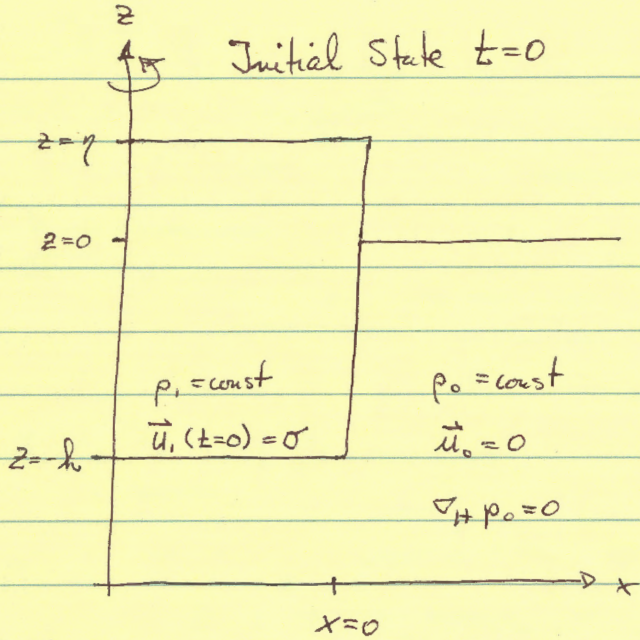
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

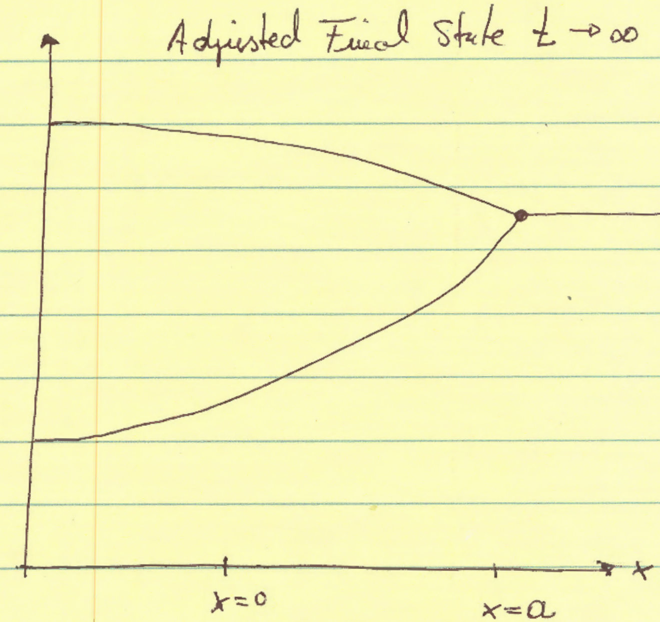
$$\frac{D}{Dt}(q) = 0 \; !$$

initial state at $t=0$

Then the potential vorticity $q = \dfrac{f + \frac{\partial v}{\partial x}}{h} = \text{const.} = \dfrac{f}{H}$ $\quad h(t=0) = H$

In the final, adjusted, steady state $\frac{\partial}{\partial t} \cdot = 0$, hence





$H = h(t=0) + \eta(t=0)$ $\qquad \nabla_H \eta = \dfrac{\rho_0 - \rho_1}{\rho_1} \nabla_H h$

but $\Delta\rho/\rho \ll 1$ & $\Delta\eta/\Delta h \ll 1$

In upper layer

$$p_1(x,y,z,t) = \cancel{p_a} + \rho_1 g(\eta - z)$$

$$\nabla_H p_1 = \rho_1 g \nabla_H \eta = \rho_1 g \frac{\rho_0 - \rho_1}{\rho_1} \nabla_H h$$

$$= g(\rho_0 - \rho_1) \nabla_H h$$

or $\dfrac{1}{\rho_0} \nabla_H p_1 = g' \nabla_H h$ $\qquad$ where $g' = \dfrac{\rho_0 - \rho_1}{\rho_0} g$

reduced gravity

In the final "adjusted" state, there is no time-dependence, and we have:

x-momentum $u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}$ — geostrophy of along-front velocity

y-momentum $u\frac{\partial v}{\partial x} + fu = 0$ — $u = 0$ → actually $u\left(\frac{\partial v}{\partial x} + f\right) = 0$

continuity $0 = 0$ — degenerate, two independent equations for three variables (u, v, h)

→ $u \cdot h = \text{const}\big|_{\text{all } x}$

there is an x where $h=0$ ⇒ $u=0$

$h(t=0)\big|_{x<0} = H$

$h(t) \neq H$

but vorticity of geostrophic velocity v to the rescue (as usual)

$$q = \frac{f + \frac{\partial v}{\partial x}}{h} = \boxed{\frac{f + \frac{g'}{f}\frac{\partial^2 h}{\partial x^2}}{h} = \text{const} = \frac{f}{H}}$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 h}{\partial x^2} - \frac{f^2}{g'H}h = -\frac{f^2}{g'}}$$

note $c_i^2 = g'H$

$\frac{c_i}{f} = L_D$ internal Rossby radius

which has solutions

$$\boxed{\begin{aligned} h(x) &= H\left(1 - e^{(x-a)/L_D}\right) \\ v(x) &= -\sqrt{g'H}\, e^{(x-a)/L_D} \end{aligned}}$$

x —> a

→ h(x) = 0

→ v(x) = -sqrt(g'*h)

where a is the unknown location of the front (where $h = 0$)

Geostrophic jet of uniform potential vorticity q was first developed by Stommel (1957) as a model for the Gulfstream

volume conservation gives $\int_{-\infty}^{0} (H - h)\,dx = \int_{0}^{a} h\,dx \quad \Rightarrow \quad a = L_D$