In the final "adjusted" state, there is no time-dependence, and we have:

x-momentum $u\frac{\partial u}{\partial x} - fv = -g'\frac{\partial h}{\partial x}$ — geostrophy of along-front velocity

y-momentum $u\frac{\partial v}{\partial x} + fu = 0$ — $u = 0$ → actually $u\left(\frac{\partial v}{\partial x} + f\right) = 0$

continuity $0 = 0$ — degenerate, two independent equations for three variables (u, v, h)

→ $u \cdot h = \text{const}\big|_{\text{all } x}$

there is an x where $h=0$ ⇒ $u=0$

but vorticity of geostrophic velocity v to the rescue (as usual)

$x<0$: $h(t\to 0)\big| = H$, $h(t) \neq H$

$$q = \frac{f + \frac{\partial v}{\partial x}}{h} = \boxed{\frac{f + \frac{g'}{f}\frac{\partial^2 h}{\partial x^2}}{h} = \text{const} = \frac{f}{H}}$$

&

$$\boxed{\frac{\partial^2 h}{\partial x^2} - \frac{f^2}{g'H}\,h = -\frac{f^2}{g'}}$$

note $c_i^2 = g'H$, $\frac{c_i}{f} = L_D$ internal Rossby radius

which has solutions

$$\boxed{\begin{aligned} h(x) &= H\left(1 - e^{(x-a)/L_D}\right) \\ v(x) &= -\sqrt{g'H}\; e^{(x-a)/L_D} \end{aligned}}$$

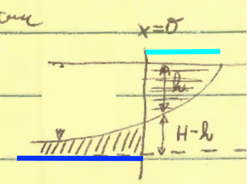
$x \to a$: → $h(x) = 0$

→ v(x) = -sqrt(g'*h)

where a is the unknown location of the front (where $h = 0$)

Geostrophic jet of uniform potential vorticity q was first developed by Stommel (1957) as a model for the Gulfstream

volume conservation gives $\int_{-\infty}^{0} (H-h)\,dx = \int_0^a h\,dx$ ⇒ $a = L_D$



Note that

$$\frac{f}{H} = \frac{f+\zeta}{h} \xrightarrow[h\to 0]{x\to a} \frac{0}{0} \quad \text{implies } \zeta \to -f$$

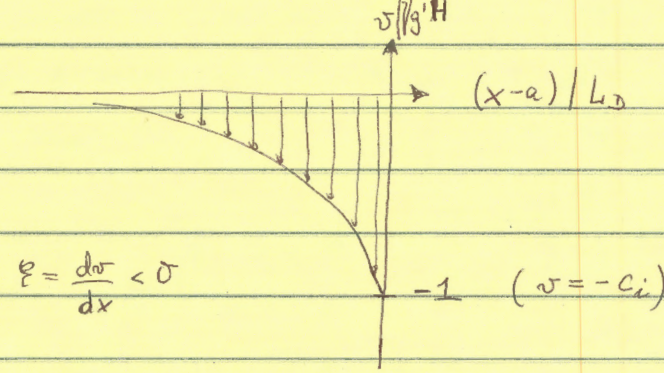
frontal dynamics $h \to 0$

out-cropping isopycnals

hence

$$R_o = \frac{|\zeta|}{f} \longrightarrow 1$$

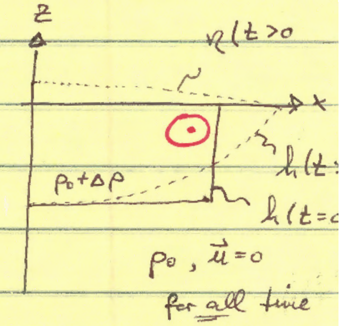
Rossby Number Ro = O(1)
nonlinear advection important



$\zeta = \dfrac{dv}{dx} < 0$

anti-cyclonic (clockwise)

shear (relative vorticity)



the geostrophically balanced flow here is out-of-page $(v<0)$

The volume transport

$$V = \int_{-\infty}^{L_D} h\cdot v \, dx = \frac{-H\sqrt{g'H}\cdot L_D}{2} = -\frac{g'H^2}{2f} \qquad \frac{m}{s^2}\cdot\frac{m^2}{s^{-1}} =$$

$$= \frac{(-g'H)/f/H}{2}$$

initial state potential energy $= \dfrac{1}{2}\rho_0 \displaystyle\int_{-\infty}^{0} g' H^2 \, dx = APE_i$

final state kinetic energy $= \dfrac{1}{2}\rho_0 \displaystyle\int_{-\infty}^{L_D} h v^2 \, dx = KE_f$

final state potential energy $= \dfrac{1}{2}\rho_0 \displaystyle\int_{-\infty}^{L_D} g' h^2 \, dx = APE_f$

$$\left.\begin{aligned}\Delta APE &= APE_i - APE_f = \frac{3}{12}\, g' H^2 \cdot L_D \\ \Delta KE &= KE_f - \underset{0}{\cancel{KE_i}} = \frac{1}{12}\, g' H^2 \cdot L_D\end{aligned}\right\}\quad \Delta KE = 3 \cdot \Delta APE$$

Only 1/3 of the available potential energy is converted to kinetic energy of the geostrophic state. Where did the other 2/3 go?

Transient wave motions $\longrightarrow$ internal Poincare waves traveling on the density interface

Without proof: The adjusted geostrophic state is the state of minimum energy