Fully non-linear equations of motion for (u,v,w,p, and rho')

$$\frac{du}{dt} - f_0 v - \beta_0 y v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2}$$ x-momentum

$$\frac{dv}{dt} + f_0 u - \beta_0 y u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v}{\partial y^2}$$ y-momentum

$$0 = -\frac{\partial p}{\partial z} - \rho' g$$ z-momentum

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ continuity
volume conservation

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$ density equation
(without diffusion)

With $\boxed{Ro_T \sim Fr^2 \sim E_v \sim \beta \sim R_o \ll 1}$ [we will return to this later again]

non-dimensional parameters

and

O(1) O(Ro) O(Ro^2)

$$u = U\left[u_0^* + R_o u_1^* + O(R_o^2)\right] \qquad u_0 = U \cdot u_0^* = O(U)$$
$$p = P\left[p_0^* + R_o p_1^* + O(R_o^2)\right] \qquad u_1 = U \cdot u_1^* = O(R_o U)$$

Scale non-dimensional variable m/s = m/s * nd

"nd" is no dimension & O(1)

The $O(1)$ equations are geostrophic

O(1) velocities

$$v_0 = \frac{1}{\rho_0 f_0}\frac{\partial p_0}{\partial x}, \quad u_0 = -\frac{1}{\rho_0 f_0}\frac{\partial p_0}{\partial y}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 = \frac{\partial w_0}{\partial z}$$

2 independent equations
for three variables
—> p0 still "unknown"

Order Ro or O(Ro) with known O(1) variables (u0,v0,p0)
but unknown O(Ro) variables (u1,v1,p1)

Then the full ageostrophic x-momentum written in terms of geostrophic velocity or pressure becomes

$$\underbrace{-\frac{1}{\rho_0 f_0}\frac{\partial}{\partial y}\frac{\partial p_0}{\partial t}}_{O(Ro_T)} \underbrace{-\frac{1}{\rho_0^2 f_0^2} J\left(p_0, \frac{\partial p_0}{\partial y}\right)}_{O(Ro)} \gg \underbrace{-\frac{1}{\rho_0 f_0} w\frac{\partial^2 p_0}{\partial y \partial z}}_{O(Ro \cdot Fr^2)} - \underbrace{\boxed{f_0 v_1}}_{O(Ro)} \underbrace{-\frac{\beta_0}{\rho_0 f_0} y \frac{\partial p_0}{\partial y}}_{O(\beta)} =$$

from $\frac{\partial u}{\partial t}$ $\quad$ $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ ~~from~~ $\quad$ $w\frac{\partial u}{\partial z}$

Jacobian operator:

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} = -J(b,a)$$

e.g.:

$$\vec{u}_H \cdot \nabla_H = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

as in

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{(\rho_0 f_0)^2} J\left(p, \frac{\partial p}{\partial y}\right)$$

$$= \underbrace{\boxed{-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}}}_{O(Ro)} \underbrace{-\frac{\nu}{\rho_0 f_0}\frac{\partial^3 p_0}{\partial y \partial^2 z}}_{O(E_v)}$$

Or @ $O(Ro, \beta, E_v)$:

$$\boxed{\underbrace{v_1}_{\text{unknown velocity}} = \underbrace{+\frac{1}{\rho_0 f_0^2}\frac{\partial p_1}{\partial x}}_{\text{unknown pressure gradient}} \underbrace{-\frac{1}{\rho_0 f_0^2}\frac{\partial^2 p_0}{\partial y \partial t} \cdots}_{\text{known ageostrophic contributions that are ALL known if } p_0 \text{ is known}}}$$

~~O(Ro)~~)

$$\boxed{u_1 = -\frac{1}{\rho_0 f^2}\frac{\partial p_1}{\partial y} - \cdots}$$

Put these (u_1, v_1) into continuity

$$-\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = \boxed{\frac{\partial w_1}{\partial z} = +\frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p_0 + J\left(p_0, \nabla^2 p_0\right) + \beta_0 \frac{\partial p_0}{\partial x} - \nu \nabla^2\left(\frac{\partial^2 p_0}{\partial z^2}\right)\right]}$$

The $\frac{\partial p_1}{\partial x}$ and $\frac{\partial p_1}{\partial y}$ terms dropped out, as we basically took $\frac{\partial}{\partial x}$ (y-mom) $- \frac{\partial}{\partial y}$ (x-mom)

that is, we can interpret this as a vorticity equation that includes a new variable, an ageostrophic vertical velocity w_1 in addition to the also unknown geostrophic pressure p_0 $\longrightarrow$ need another equation to close the problem:

$$\frac{\partial \rho'}{\partial t} + \underbrace{\frac{1}{\rho_0 f_0} J\left(p_0, \rho'\right)}_{= u_0 \frac{\partial \rho'}{\partial x} + v_0 \frac{\partial \rho'}{\partial y}} - \underbrace{\frac{\rho_0 N^2}{g} w_1}_{w_1 \frac{\partial \bar\rho}{\partial z}} = 0$$

$$\boxed{N^2 = -\frac{g}{\rho_0}\frac{\partial \bar\rho}{\partial z} = N^2(z)}$$

known function of z

background stratification

or $\rho' = -\frac{1}{g}\frac{\partial p_0}{\partial z}$

w1 = w1(p0):

$$w_1 = \frac{g}{N^2 \rho_0}\left[-\frac{1}{g}\frac{\partial}{\partial t}\left(\frac{\partial p_0}{\partial z}\right) + \frac{1}{\rho_0 f_0} J\left(p_0, -\frac{1}{g}\frac{\partial p_0}{\partial z}\right)\right]$$

We now have

w_1 expressed in terms of geostrophic pressure p_0 from the density equation

We also have

$\frac{\partial w_1}{\partial z}$ expressed in terms of geostrophic pressure p_0 from the vorticity equation

$$\left.\frac{\partial w_1}{\partial z}\right|_{\text{density}} = \left.\frac{\partial w_1}{\partial z}\right|_{\text{vorticity}}$$ gives

Quasi-Geostrophic Vorticity Equation

$$\boxed{\frac{\partial}{\partial t}\left[\underline{\nabla^2 p_0 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p_0}{\partial z}\right)}\right] + \frac{1}{\rho_0 f_0} J\left[p_0, \underline{\nabla^2 p_0 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p_0}{\partial z}\right)}\right] + \beta_0 \frac{\partial p_0}{\partial x} = \nu \frac{\partial^2}{\partial z^2}\left(\nabla^2 p_0\right)}$$

gives geostrophic pressure p0=p0(x,y,z,t)

Single PDE for a continuously stratified fluid under the QG (quasi-geostrophic) assumption that isopycnal excursions ρ' from a basic state $\bar{\rho}$ are small, that is,

$$\rho = \bar{\rho}(z) + \rho' \qquad \text{with} \qquad \rho' \ll \bar{\rho}$$

Recall geostrophic velocities and their pressure gradients

$$\frac{\partial p_0}{\partial x} = \rho_0 f_0 v_0 \qquad \frac{\partial p_0}{\partial y} = -\rho_0 f_0 u_0$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

allows

$$u_0 = -\frac{\partial \psi}{\partial y} \qquad v_0 = +\frac{\partial \psi}{\partial x} \qquad \text{streamfunctions}$$

$\Downarrow$

$$\boxed{p_0 = \rho_0 f_0 \psi \qquad \text{geostrophic pressure is proportional to } \psi}$$

Also

$$\frac{\partial p_0}{\partial z} = -g\rho'$$

becomes

$$\rho_0 f_0 \frac{\partial \psi}{\partial z} = -g\rho' \qquad \text{or} \qquad \frac{\partial \psi}{\partial z} = -\frac{g}{\rho_0 f_0}\rho'$$

Furthermore, the prominent term

$$\nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y \equiv q$$

potential vorticity

$$\frac{\partial q}{\partial t} + J(\psi, q) = \nu \frac{\partial^2}{\partial z^2}\left(\nabla^2 \psi\right)$$

$$\frac{D_0}{Dt}\left(q\right) = \nu \frac{\partial^2}{\partial z}\left(\nabla^2 \psi\right)$$

geostrophic

where $\frac{D_0 \cdot}{Dt} \equiv \frac{\partial \cdot}{\partial t} + u_0 \frac{\partial \cdot}{\partial x} + v_0 \frac{\partial \cdot}{\partial y}$

$$\frac{D_0}{Dt}\left[\nabla^2 \psi + \beta_0 y + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] = \nu \frac{\partial^2}{\partial z^2}\left(\nabla^2 \psi\right)$$

relative vorticity of the geostrophic flow

vortex tube stretching due to stratification

diffusion of relative vorticity of the geostrophic flow

Observations:

1. How to reconcile with $\frac{D}{Dt}\left(\frac{f + \xi}{h}\right) = 0$?

$\bar{h} \gg h'$

For each "layer" of thickness $h = \bar{h} + h' = \bar{h} + \frac{\Delta \rho}{\partial \bar{\rho}/\partial z}$

$$q = \frac{f + \xi}{h} = \frac{f_0 + \Delta f + \xi}{\bar{h} + h'} \simeq \frac{1}{\bar{h}}\left(f_0 + \Delta f + \xi - \frac{f_0}{\bar{h}} \cdot h' + \dots\right)$$

$$= q_0 + \frac{q'}{\bar{h}}$$

where $q_0 = \frac{f_0}{\bar{h}}$ and $q' = \underbrace{\Delta f}_{\beta y} + \underbrace{\xi}_{\nabla^2 \psi} - \underbrace{\frac{f_0}{\bar{h}} \cdot h'}_{\frac{f_0^2}{N^2}\frac{\partial^2 \psi}{\partial z^2}} = PV_{QG}$

O(1) effects

O(Ro) effects
Ro << 1