Fully non-linear equations of motion for (u,v,w,p, and rho')

$$\frac{du}{dt} - f_0 v - \beta_0 y v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2}$$ x-momentum

$$\frac{dv}{dt} + f_0 u - \beta_0 y u = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v}{\partial y^2}$$ y-momentum

$$0 = -\frac{\partial p}{\partial z} - \rho' g$$ z-momentum

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ continuity
volume conservation

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} = 0$$ density equation
(without diffusion)

With $\boxed{Ro_T \sim Fr^2 \sim E_v \sim \beta \sim Ro \ll 1}$ [we will return to this later again]

non-dimensional parameters

and

O(1) O(Ro) O(Ro^2)

$$u = U\left[u_0^* + Ro\, u_1^* + O(Ro^2)\right] \qquad u_0 = U \cdot u_0^* = O(U)$$

$$p = P\left[p_0^* + Ro\, p_1^* + O(Ro^2)\right] \qquad u_1 = U \cdot u_1^* = O(Ro\, U)$$

Scale non-dimensional variable m/s = m/s * nd

The $O(1)$ equations are geostrophic

"nd" is no dimension & O(1)

O(1) velocities

$$v_0 = \frac{1}{\rho_0 f_0}\frac{\partial p_0}{\partial x}, \qquad u_0 = -\frac{1}{\rho_0 f_0}\frac{\partial p_0}{\partial y}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 = \frac{\partial w_0}{\partial z}$$

2 independent equations
for three variables
—> p0 still "unknown"

Order Ro or O(Ro) with known O(1) variables (u0,v0,p0)
but unknown O(Ro) variables (u1,v1,p1)

Then the full ageostrophic x-momentum written in terms of geostrophic velocity or pressure becomes

$$\underbrace{-\frac{1}{\rho_0 f_0}\frac{\partial}{\partial y}\frac{\partial p_0}{\partial t}}_{O(Ro_T)} \underbrace{-\frac{1}{\rho_0^2 f_0^2} J\left(p_0, \frac{\partial p_0}{\partial y}\right)}_{O(Ro)} \gg \underbrace{-\frac{1}{\rho_0 f_0}\, w\frac{\partial^2 p_0}{\partial y\,\partial z}}_{O(Ro\cdot Fr^2)} \underbrace{-\boxed{f_0 v_1}}_{O(Ro)} \underbrace{-\frac{\beta_0 y}{\rho_0 f_0}\frac{\partial p_0}{\partial y}}_{O(\beta)} =$$

from $\frac{\partial u}{\partial t}$ $\qquad$ $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$ ~~from~~ $\qquad$ $w\frac{\partial u}{\partial z}$

Jacobian operator:

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} = -J(b,a)$$

e.g.:

$$\vec{u}_H\cdot\nabla_H = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

as in

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{(\rho_0 f_0)^2} J\left(p, \frac{\partial p}{\partial y}\right)$$

$$= \underbrace{\boxed{-\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}}}_{O(Ro)} - \underbrace{\frac{\nu}{\rho_0 f_0}\frac{\partial^3 p_0}{\partial y\,\partial^2 z}}_{O(E_v)}$$

Or @ $O(Ro, \beta, E_v)$:

$$\underbrace{v_1}_{\substack{\text{unknown velocity}\\ \cancel{O(Ro)}}} = \underbrace{+\frac{1}{\rho_0 f_0^2}\frac{\partial p_1}{\partial x}}_{\substack{\text{unknown}\\ \text{pressure gradient}}} \underbrace{-\frac{1}{\rho_0 f_0^2}\frac{\partial^2 p_0}{\partial y\,\partial t} \cdots}_{\substack{\text{known ageostrophic contributions}\\ \text{that are ALL known if } p_0 \text{ is known}}}$$

$$\overbrace{u_1} = \overbrace{-\frac{1}{\rho_0 f^2}\frac{\partial p_1}{\partial y}} \overbrace{- \cdots}$$