We now have

$w_1$ expressed in terms of geostrophic pressure $p_0$ from the density equation

We also have

$\dfrac{\partial w_1}{\partial z}$ expressed in terms of geostrophic pressure $p_0$ from the vorticity equation

$$\left.\frac{\partial w_1}{\partial z}\right|_{\text{density}} = \left.\frac{\partial w_1}{\partial z}\right|_{\text{vorticity}} \qquad \text{gives}$$

Quasi-Geostrophic Vorticity Equation (fully nonlinear)

$$\boxed{\frac{\partial}{\partial t}\left[\underline{\nabla^2 p_0 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p_0}{\partial z}\right)}\right] + \frac{1}{\rho_0 f_0} J\left[p_0 , \underline{\nabla^2 p_0 + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p_0}{\partial z}\right)}\right] + \beta_0 \frac{\partial p_0}{\partial x} = \nu \frac{\partial^2}{\partial z^2}\left(\nabla^2 p_0\right)}$$

gives geostrophic pressure p0=p0(x,y,z,t)

Single PDE for a continuously stratified fluid under the QG (quasi-geostrophic) assumption that isopycnal excursions $\rho'$ from a basic state $\bar{\rho}$ are small, that is,

$$\rho = \bar{\rho}(z) + \rho' \qquad \text{with} \qquad \rho' \ll \bar{\rho}$$

Recall geostrophic velocities and their pressure gradients

$$\frac{\partial p_0}{\partial x} = \rho_0 f_0 v_0 \qquad \frac{\partial p_0}{\partial y} = -\rho_0 f_0 u_0$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

allows

$$u_0 = -\frac{\partial \psi}{\partial y} \qquad v_0 = +\frac{\partial \psi}{\partial x} \qquad \text{stream functions}$$

$\Downarrow$

$$\boxed{p_0 = \rho_0 f_0 \psi \qquad \text{geostrophic pressure is proportional to } \psi}$$

Also

$$\frac{\partial p_0}{\partial z} = -g\rho'$$

becomes

$$\rho_0 f_0 \frac{\partial \psi}{\partial z} = -g\rho' \qquad \text{or} \qquad \frac{\partial \psi}{\partial z} = -\frac{g}{\rho_0 f_0}\rho'$$

Furthermore, the prominent term

$$\nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y \equiv q$$

potential vorticity

$$\frac{\partial q}{\partial t} + J(\psi, q) = \nu \frac{\partial^2}{\partial z^2}(\nabla^2 \psi)$$

$$\frac{D_o}{Dt}(q) = \nu \frac{\partial^2}{\partial z}(\nabla^2 \psi)$$

geostrophic

where $\frac{D_o \cdot}{Dt} = \frac{\partial \cdot}{\partial t} + u_o \frac{\partial}{\partial x} + v_o \frac{\partial}{\partial y} \cdot$

$$\boxed{\frac{D_o}{Dt}\left[\nabla^2\psi + \beta_o y + \frac{\partial}{\partial z}\left(\frac{f_o^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] = \nu \frac{\partial^2}{\partial z^2}(\nabla^2\psi)}$$

relative vorticity of the geostrophic flow

vortex tube stretching due to stratification

diffusion of relative vorticity of the geostrophic flow

Observations:

1. How to reconcile with $\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = 0$ ?

$\bar{h} \gg h'$

For each "layer" of thickness $h = \bar{h} + h' = \bar{h} + \frac{\Delta\rho}{\partial\bar{\rho}/\partial z}$

H >> h'

$$q = \frac{f+\xi}{h} = \frac{f_o + \Delta f + \xi}{\bar{h} + h'} \simeq \frac{1}{\bar{h}}\left(\underline{f_o + \Delta f + \xi} - \underline{\frac{f_o}{\bar{h}} \cdot h'} + \dots\right)$$

Taylor Series of q(h) @ h=H q(h) = q(H) - q(H)*h'/H + O(h'/H)^2

$$= q_o + \frac{q'}{\bar{h}}$$

where $q_o = \frac{f_o}{\bar{h}}$ and $q' = \underbrace{\Delta f}_{\beta y} + \underbrace{\xi}_{\nabla^2\psi} - \underbrace{\frac{f_o}{\bar{h}} \cdot h'}_{\frac{f_o^2}{N^2}\frac{\partial^2\psi}{\partial z^2}} = PV_{QG}$

O(1) effects

O(Ro) effects
Ro << 1

Compare $\xi = \nabla^2 \psi \sim U/L$ the relative vorticity xi

with the stretching term due to stratification

$$\frac{\partial}{\partial z} \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \sim \frac{f_0^2}{N^2} \frac{U \cdot L}{H^2} :$$

u = - d(xi)/dy
d(xi) ~ u*dy ~ U * L

$$\frac{\text{relative vorticity}}{\text{stretching}} \sim \frac{U/L}{f_0^2/N^2 \cdot U \cdot L/H^2} = \frac{N^2 H^2}{f_0^2 L^2} \equiv \text{Burger \#} = \left(\frac{L_D}{L}\right)^2$$

ratio of
two length-scales

Case-1 : $Bu \ll 1$ : stretching due to stratification dominates
$L \gg L_D$
which was also the limit for $\rho \approx \rho_0$ and
nearly geostrophic flow with topography along f/H contours

Case-2 : $Bu \gg 1$ : relative vorticity dominates over stratification
$L \ll L_D$
flow in different layers not coupled stiff fluid, little stretchin

Case-3 : $Bu = O(1)$ : relative vorticity and stretching of the same order
flow in different layers are coupled

$$\frac{D_0}{Dt}\underbrace{\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + \beta_0 y\right]}_{q \,=\, \text{potential vorticity}} = \nu^2\frac{\partial^2}{\partial z^2}\left(\nabla^2\psi\right)$$ (120A)

Nonlinear equation !

$$\frac{\partial q}{\partial t} + \underbrace{J(\psi, q)}_{\text{nonlinear}} = \text{diffusio}$$

jump to p.-124 for linear versio

Rossby # $\quad R_o = U/fL$

Froude # $\quad F_r = U/NH$

Burger # $\quad B_u = \left(\frac{NH}{fL}\right)^2 = \left(\frac{R_o}{F_r}\right)^2 = \left(\frac{L_D}{L}\right)^2$

$L_D = \frac{\sqrt{g'H}}{f}$

ratio of $\frac{\text{internal deformation radius}}{\text{length scale of motion}}$

ratio of $\frac{\text{relative vorticity}}{\text{stretching due to stratification}}$

$B_u \ll 1$ : $R_o \ll F_r$ or $L_D \ll L$ or $\xi \ll$ stretching — behaves like barotro[pic] flow

$B_u \gg 1$ : $R_o \gg F_r$ or $L_D \gg L$ or $\xi \gg$ stretching — strongly stratified flow that "layers" are so stiff, they do not interact

$B_u \sim 1$ : $R_o \sim F_r$